NEDERL. AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLIV

No. 9

President: J. VAN DER HOEVE
Secretary: M. W. WOERDEMAN

CONTENTS

BURGERS, J. M.: "On the influence of the concentration of a suspension upon the sedimentation velocity (in particular for a suspension of spherical particles)", p. 1045.

WEITZENBÖCK, R., und W. J. BOS: "Zur projektiven Differentialgeometrie der Regelflächen im R_4". (Sechste Mitteilung), p. 1052.

GODEAUX, LUCIEN: "Une propriété des variétés de VERONESE". (Communicated by Prof. W. VAN DER WOUDE), p. 1058.

MEIJER, C. S.: "Multiplikationstheoreme für die Funktion $G_{p,q}^{m,n}(z)$". (Communicated by Prof. J. G. VAN DER CORPUT), p. 1062.

REMAK, ROBERT: "Ein Satz über die sukzessiven Minima bei definiten quadratischen Formen". (Communicated by Prof. J. G. VAN DER CORPUT), p. 1071.

VEEN, S. C. VAN: "Stark konvergente Entwicklungen für die vollständigen elliptischen Integrale erster und zweiter Art". II. (Communicated by Prof. J. G. VAN DER CORPUT), p. 1077.

VEEN, S. C. VAN: "Ueber die Entwicklung der unvollständigen elliptischen Integrale erster und zweiter Art in stark konvergenten Reihen". II. (Communicated by Prof. J. G. VAN DER CORPUT), p. 1085.

RUTGERS, J. G.: "Sur des séries et des intégrales définies contenantes les fonctions de BESSEL". VI. (Communicated by Prof. J. A. SCHOUTEN), p. 1092.

BUNGENBERG DE JONG, H. G., and E. G. HOSKAM: "Motory phenomena in coacervate drops in a diffusion field and in the electric field". (Communicated by Prof. J. VAN DER HOEVE), p. 1099.

BUNGENBERG DE JONG, H. G., E. G. HOSKAM and L. H. V. D. BRANDHOF-SCHAEGEN: "The influence of the temperature on the complexcoacervation of gelatine and gum arabic". (Communicated by Prof. J. VAN DER HOEVE), p. 1104.

GORTER, A. and W. NADORT: "Composition of Gas in the Intercellular Spaces of Potatoes". (Communicated by Prof. J. C. SCHOUTE), p. 1112.

DILLEWIJN, C. VAN: "Some technical remarks about the colchicine treatment of Gramineae". (Communicated by Prof. A. H. BLAAUW). (With one plate), p. 1118.

K 244

FUNKE, G. L.: "The growth of the petioles of waterplants in solutions of phytohormones". I. (Communicated by Prof. L. G. M. BAAS BECKING), p. 1121.

BOK, R.: "The influence of oxygen-nitrogen mixtures upon the dwarfing of Ardisia Crispa (Thunb.) A. DC". (Communicated by Prof. L. G. M. BAAS BECKING), p. 1128.

WESTERVELD, J.: "Three geological sections across South Sumatra". (Communicated by Prof. H. A. BROUWER). (With one karte), p. 1131.

BELLEN, R. C. VAN, J. F. C. DE WITT PUYT, A. C. RUTGERS, and J. VAN SOEST: "Smaller Foraminifera from the Lower Oligocene of Cuba". (Communicated by Prof. L. RUTTEN). (With one plate and one table), p. 1140.

HOOIJER, D. A.; "Note on a diseased dental condition in Hippopotamus amphibius L". (Communicated by Prof. L. RUTTEN). (With one plate), p. 1147.

POSTMA, N.: "Ueber den Tonus des Schneckenfusses (Helix pomatia L.). I. Der Tonus gemessen an der Pedalkonstanz. (Vorläufige Mitteilung). (Communicated by Prof. H. J. JORDAN), p. 1151.

BIJTEL, J.: "Some considerations concerning our knowledge of the action of the auditory organ". II. (Communicated by Prof. A. DE KLEYN), p. 1159.

ERRATA, p. 1163.

Hydrodynamics. — *On the influence of the concentration of a suspension upon the sedimentation velocity* (in particular for a suspension of spherical particles). By J. M. BURGERS. (Mededeeling No. 42 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft.)

(Communicated at the meeting of October 25, 1941.)

1. In Ch. III of the "Second Report on Viscosity and Plasticity" a few remarks have been made concerning the problem of the mutual influence of the particles of a suspension upon their velocity of sedimentation, and VON SMOLUCHOWSKI's work on this problem has been mentioned [1]). A formula giving the magnitude of the effect to be expected was not developed. In the following lines it will be attempted to supplement this point, at least for particles of spherical form.

In treating this problem several factors must be taken into account.

In the first place we must consider the magnitude of the *effective gravitational force* acting upon a single particle. When the volume of a particle is denoted by s; its density by ϱ_P; the density of the liquid by ϱ_S; the gravitational acceleration by g (in a centrifugal field $g = \omega^2 x$, ω being the angular velocity of the rotation and x the distance from the axis), this force is given by:

$$F = g\,(\varrho_P - \varrho_S)\,s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

It is often asserted that for ϱ_S in this equation we must take not the density ϱ of the pure liquid surrounding the particles, but the *mean density of the suspension* [1a]), which is given by:

$$\varrho_S = \varrho + n\,s\,(\varrho_P - \varrho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

n being the number of particles per unit of volume of the suspension. The assumption then is made that the hydrostatic force acting upon a particle will depend upon the magnitude of the *mean* hydrostatic pressure gradient, and that the latter will be equal to the weight per unit volume of the suspension.

Substitution of the expression (2) for ϱ_S into eq. (1) would give:

$$F = g\,(\varrho_P - \varrho)\,s\,(1 - n\,s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The argument that the mean hydrostatic pressure gradient will be equal to the mean weight per unit volume of the suspension, however, is not fully convincing, as for a suspension enclosed in a vessel part of the weight may be carried by the frictional forces acting at the walls. It is therefore not immediately to be seen whether the effect considered here must be superimposed upon those to be considered further on. We hope to come back to this point afterwards, and provisionally shall not go into the question whether the force F acting upon a particle is given by eq. (1) with $\varrho_S = \varrho$, or whether the correction, given by eq. (3), should be used. [1b])

---

[1]) J. M. BURGERS, Ch. III of the "Second Report on Viscosity and Plasticity", Verhand. Kon. Nederl. Akad. v. Wetenschappen (1e sectie) Vol. **16**, No. 4 (Amsterdam 1938), pp. 178/179.

[1a]) Compare *e.g.*: TH. SVEDBERG and K. O. PEDERSEN, The Ultracentrifuge (Oxford 1940), pp. 5, 47, where it is stated that — in the notation used by the authors of that work — "ϱ is the density of the solution"; and p. 59, where the specific volume $V = 1/\varrho$ of the solution is defined.

[1b]) This remark has been inserted November 8, 1941.

2. Next to the influence of the concentration of the particles upon the magnitude of the hydrostatic force, attention can be given to the circumstance that the presence of the particles brings about *an increase of the effective viscosity* of the liquid. In order to obtain a simple picture of this influence, it is convenient to assume that only one particle, say A, is acted upon by the gravitational force, the other particles temporarily being supposed to have a density equal to the density of the pure liquid. In that case these other particles will not sediment, but will be carried along by the liquid with a velocity approximately equal to the velocity which in the absence of such a particle would be found at its centre (a more accurate expression for this velocity is given by eq. (17) below). As the particles are rigid, however, they will exert an influence upon the motion of the liquid, of such nature as has been considered in Part **B** of Ch. III of the "Second Report".

In a concise and somewhat generalized form this influence can be described as follows: Every particle takes part in the translational and rotational movements of the element of volume surrounding it; owing to its rigidity, however, a particle cannot follow the deformation of this element. The particle in consequence experiences a system of forces which evidently will react upon the liquid. The system of forces with which the particle reacts upon the liquid can be represented by a system of doublets, the magnitude and orientation of which depend upon the orientation of the three principal axes of the instantaneous deformational motion of the element of volume of the liquid; upon the magnitude of the three rates of dilatation D_1, D_2, D_3 corresponding to these axes; and upon the form and the dimensions of the particle and its orientation with respect to the three axes. In the most simple case, where — due to the presence of a sufficiently intense Brownian movement — all orientations of a particle with respect to these axes are equally probable, it is sufficient to restrict to the consideration of the mean values of the doublets in the directions of the three axes. Denoting these mean values by M_1, M_2, M_3 respectively, it is found that

$$M_1 = -2k\eta D_1, \quad M_2 = -2k\eta D_2, \quad M_3 = -2k\eta D_3 \quad . \quad . \quad (4)$$

where k is a coefficient, depending upon the form and the dimensions of a particle [2]).

As there are n particles per unit of volume, there are also n of these systems of doublets per unit of volume. It will be seen that the presence of these doublets gives rise to the appearance of additional normal stresses acting through elements of surface perpendicular to the axes of the deformational field, of magnitudes respectively:

$$\sigma_1^* = -nM_1, \quad \sigma_2^* = -nM_2, \quad \sigma_3^* = -nM_3 \quad . \quad . \quad . \quad . \quad (5)$$

These stresses must be added to the ordinary viscous stresses arising in the liquid itself, which are given by the well known expressions:

$$\sigma_1 = 2\eta D_1, \quad \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

---

[2]) As the liquid is assumed to be incompressible, we have: $D_1 + D_2 + D_3 = 0$, and consequently also: $M_1 + M_2 + M_3 = 0$.

The magnitude of the components of the rotational velocity taken up by the particle can be obtained from the equations which express that the system of doublets shall have no resulting moment about the centre of the particle.

The coefficient k introduced in the text corresponds to the quantity A_{II}/s in the notation of the "Second Report", s again denoting the volume of a particle. For a spherical particle, where the question of the orientation does not present itself, $A_{II} = 5/2$ ("Second Report", p. 132); for elongated particles (cylindrical particles and rotational ellipsoids) values are given in the "Second Report" at p. 148 and pp. 151—153. Values for oblate rotational ellipsoids have been given in a paper in these Proceedings **43**, p. 314, 1940; values for some model systems consisting of a few rigidly connected spheres of equal radius *ibidem* pp. 425, 645.

Adding together both stress systems, we obtain the effective stresses, which consequently will be related to the rates of dilatation by the formulae:

$$(\sigma_1)_{\text{eff.}} = \sigma_1 + \sigma_1^* = 2\,\eta\,D_1\,(1 + nk), \quad \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence the *effective coefficient of viscosity* of the suspension is given by:

$$\eta_{\text{eff.}} = \eta\,(1 + nk) \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In all calculations referring to the motion of the suspension in bulk this effective coefficient of viscosity must be used.

When it is assumed that between the particles there act neither repulsive nor attractive forces, so that in all elements of volume surrounding the particle A mentioned above there is an equal probability of finding other particles, it seems legitimate to use the same effective coefficient also in calculating the mean field of flow, produced by the motion of the particle A under the gravitational force acting upon it. Consequently $\eta_{\text{eff.}}$ also will appear in the resistance formula for the particle A.

In such cases where repulsive forces are present it seems probable that in the resistance formula a viscosity coefficient must be used which is intermediate between $\eta_{\text{eff.}}$ as defined by (8) and η itself. This likewise is a point to which we shall return afterwards.

3. In the preceding section we had assumed that the particle A should be the only one which was subjected to the gravitational force, the other particles being supposed to have the same density as the liquid and following the motion of the latter. When now the original density is restored to these other particles, the sedimentation velocity of any one of them may be considered as the resultant of:

a) the direct influence of the gravitational force, calculated with the value $\eta_{\text{eff.}}$ for the viscosity;

b) the velocity imparted to the liquid in consequence of the fields of flow produced by all other sedimenting particles.

We therefore proceed to the calculation of this latter velocity. For this purpose we may first consider the field produced by an arbitrary particle A; next we can give to this particle all positions which it may take relatively to another particle B, and calculate the mean velocity of the flow produced at the spot of B (more accurately over the surface of B); finally this mean velocity must be multiplied by the total number of particles present.

It seems just as well possible, however, to follow a slightly different way: after having considered the field produced by an arbitrary particle A, and having found the mean velocity over the surface of B, we may give to B all positions which it can take relatively to A, the latter being kept at a fixed position. In this manner we find the average influence of A upon an arbitrary other particle; if this influence should prove to be independent of the position of A (in case it might depend upon the orientation of A, an appropriate mean value should be calculated), we again can multiply the mean velocity found in this way by the number of particles present.

Although due attention will be given to the circumstance that the particle B cannot penetrate into the particle A, no account will be taken of the fact that two or more particles B cannot at the same time be found at the same point of the field, as this would lead only to effects of the second order.

As was mentioned already in the "Second Report" it is necessary, however, to take due account of the circumstance that the liquid containing the particles is enclosed in a vessel. Every particle drags a certain amount of the liquid along with it, and as this quantity cannot pass through the walls of the vessel (nor through a free surface of the liquid, if this should be present), a return flow must occur.

It is sufficient to restrict to the consideration of the component of velocity in the direction of the gravitational force (which may be taken to be that of the x-axis); this

component will be denoted by u. We describe the field of flow produced by the particle A by the formula:

$$u = \frac{F}{8\pi\eta_{\text{eff.}}} \Phi(x, y, z; \xi, \eta, \zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where F is the force acting upon the particle A; ξ, η, ζ are the coordinates of the centre of A; x, y, z represent an arbitrary point of the field, situated e.g. upon the surface of the particle B. We write $r = \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{1/2}$. The function Φ satisfies the equation [3]):

$$\triangle\triangle\Phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

At the walls of the vessel:

$$\Phi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

At the surface of the particle A the function Φ takes a constant value, which will be denoted by Φ_0.

The exact form of Φ depends both upon the shape, the dimensions and the orientation of the particle A, and upon the dimensions of the vessel. In the case of a *spherical particle* of radius a we have, for small values of r:

$$\Phi = \frac{1}{r} + \frac{(x-\xi)^2}{r^3} + \frac{a^2}{3r^3} - \frac{a^2(x-\xi)^2}{r^5} + \chi \quad . \quad . \quad . \quad . \quad (12)$$

where χ remains finite for $r = 0$, and represents the influence of the walls of the vessel. By way of example it may be mentioned that for a spherical particle situated at the centre of a spherical vessel with radius R, large in comparison with a, we should have, approximately:

$$\chi = -\frac{3}{R} + \frac{2r^2 - (x-\xi)^2}{R^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

terms of the order a^2/R^3 being neglected.

In this case the value of Φ at the surface of the particle becomes:

$$\Phi_0 = 4/3a - 3/R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In the cases to be considered here, however, the influence of the term denoted by χ is of importance only in integrals extended over the whole space taken in by the system, such as occur in the deduction of eq. (19) below and in that of eq. (22). In consequence of the large value of the ratio of the dimensions of the vessel to those of the particles χ generally can be neglected in calculations referring to the surface of the particle or to its

---

[3]) The relations (10) and (11) immediately follow from the equations of motion, which upon neglecting the inertia terms take the form:

$$\partial p/\partial x = \eta\triangle u; \quad \partial p/\partial y = \eta\triangle v; \quad \partial p/\partial z = \eta\triangle w;$$

with

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0.$$

Hence we have:

$$\triangle p = 0, \quad \triangle\triangle u = 0, \text{ etc.}$$

At the walls of the vessel $u = v = w = 0$.

The influence of OSEEN's correction of STOKES' formula, if necessary, can be assumed to have been taken up in the term denoted by χ in eq. (12); it does not affect the form of Φ for small values of r.

immediate neighbourhood. Hence for spherical particles we may write, with sufficient approximation for our purpose:

$$\Phi_0 = 4/3a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

4. In order to find the influence of this field upon the particle B we calculate the mean value of the velocity over the surface of B. This procedure in many cases is sufficiently accurate for obtaining the influence of A upon B; in the case of spherical particles the error remains small even when the two spheres are in contact [4]).

The mean value of Φ over the surface S of the particle B is defined by the integral:

$$(1/S) \int\int dS\, \Phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If we assume that the particle B may take all orientations in space with equal probability, we must determine the mean value of this integral for these orientations. For this purpose we consider a definite element dS of the surface, situated at a distance r_1 from the centre of the particle. Taking the average value for all orientations of the particle, the contribution by this element to the desired mean value will be:

$$(dS/S) \left(\Phi_c + \frac{1}{6} r_1^2 \triangle \Phi + \frac{1}{120} r_1^4 \triangle \triangle \Phi + \ldots \right). \quad . \quad . \quad (16)$$

where Φ_c is the value of Φ at the centre of the particle B. As $\triangle\triangle\Phi = 0$, we may restrict this development to the first and the second terms. Hence the mean value (to be denoted by Φ_m) of Φ over the surface of the particle B, for all orientations of the latter, becomes:

$$\Phi_m = \Phi_c + \frac{1}{6} \triangle \Phi \,.\, (1/S) \int\int dS\, r_1^2 = \Phi_c + \frac{1}{6} r_0^2 \triangle \Phi. \quad . \quad (17)$$

where r_0^2 represents the mean value of r_1^2 over the surface S (radius of gyration of S with respect to the centre).

In the case of a sphere of radius a we have: $r_0 = a$.

---

[4]) In the case of two spheres, both of radius a, in an unlimited field, the application of formula (17) of the text below gives:

$$\Phi_m = (1 + \cos^2 \vartheta)/r + 2a^2 (1 - 3 \cos^2 \vartheta)/3 r^3,$$

where $\cos \vartheta = (x - \xi)/r$. When both spheres are acted upon by a force F in the same direction, the velocity acquired by each sphere is given by:

$$U = \frac{F}{8 \pi \eta} (\Phi_0 + \Phi_m) = \frac{F}{6 \pi \eta a} (1 + \tfrac{3}{4} a \Phi_m).$$

A few values of the factor $m = 1 + 3a \Phi_m/4$, for the case $\vartheta = 0$, are collected in the following table, together with those of $1/m$ and values of a coefficient λ, calculated with the aid of more precise methods by other authors:

r/a	m	$1/m$	λ
4.704	1.309	0.764	0.768
3.086	1.452	0.689	0.702
2.256	1.577	0.634	0.663
2	1.625	0.616	0.645

Compare: C. W. OSEEN, Hydrodynamik (Leipzig 1927), p. 162.

5. We now must calculate the mean value of Φ_m for all positions of the particle B with respect to A. Provisionally we assume that the orientation in space of A is kept unchanged. The space around A will be divided into three regions:

(I) a layer G^* of thickness r_{max} along the walls of the vessel;

(II) the region C^* between a spherical surface C with radius $2r_{max}$ around the centre of the particle A and the surface S of the latter;

(III) the remaining space, to be denoted by G, between the layer along the wall and the spherical surface C.

Here r_{max} will denote the maximum value of the distance r_1 of any element dS of the surface S from the centre of the particle. It is only when the centre of the particle B is situated in G, that free rotation, both of A and of B, is possible, so that we may apply the assumption of equal probability of all spatial orientations.

In the layer along the wall Φ will be at most of the order $r_{max} R^{-2}$, if R is a measure for the dimensions of the vessel.

The contribution by the region G to the mean value sought, and those by the regions C^* and G^* (in which regions the geometrical relations are far more complicated) must be treated separately.

In the case of spherical particles where r_{max} is equal to the radius a of the particle, no particle can have its centre in the regions C^* and G^*. The contributions of these regions then wholly disappear.

6. The contribution by the region G is given by:

$$\left.\begin{aligned}(1/\Omega)\iiint dx\,dy\,dz\;\Phi_m &= \\ &= (1/\Omega)\left(\iiint dx\,dy\,dz\;\Phi_c + \frac{1}{6}r_0^2\iiint dx\,dy\,dz\;\triangle\,\Phi\right)\end{aligned}\right\}. \quad (18)$$

Ω being the volume of the vessel.

Now it follows from the equation of continuity that for any plane $x = \text{const.}$:

$\iint dydz\;\Phi_c = 0$, if this plane does not cut the particle A;

$\iint dydz\;\Phi_c = -\Phi_0\,\Sigma$, if this plane cuts the particle A, where Σ is the area of the cut, while Φ_0 is the value of Φ at the surface of the particle, which is equal to the velocity of the latter, multiplied by $8\pi\eta_{\text{eff.}}/F$ (it is sufficiently accurate here to take the value of the velocity uncorrected for the influence of the other particles). Hence, when the integration is extended over the whole region between the surface of the particle A and the walls of the vessel:

$$\iiint dx\,dy\,dz\;\Phi_c = -\Phi_0\,s \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

s, as before, being the volume of the particle.

When the integration is extended over the region G only, we must exclude the domains C^* and G^*, and obtain:

$$\iiint_G dx\,dy\,dz\;\Phi_c = -\Phi_0\,s - \iiint_{C^*} dxdy\,dz\;\Phi_c - \iiint_{G^*} dxdy\,dz\,\Phi_c \quad (20)$$

The calculation should be performed for all orientations of A. In the first term on the right hand side we consequently may take Φ_0 to represent the mean value for all orientations. The mean value of the second term for shortness will be written $k_1 r_0^2$, that

of the third one $-k_2 r_0^2$ (k_1 and k_2 being numerical coefficients). In this way we have:

$$\iiint_G dx\,dy\,dz\,\Phi_c = -\Phi_0\, s - k_1 r_0^2 + k_2 r_0^2 \quad . \quad . \quad . \quad (21)$$

Next considering the second integral of the right hand member of (18), we have:

$$\iiint_G dx dy dz\,\triangle\Phi = \iint dS_e\,\partial\Phi/\partial n - \iint dS_C\,\partial\Phi/\partial r \quad . \quad . \quad (22)$$

where dS_e represents an element of the exterior surface of G (*i.e.* of the wall of the vessel), and dS_C an element of the spherical surface C. Both surface integrals appear to give numerical coefficients, to be denoted by k_3, $-k_4$ respectively, so that:

$$\iiint_G dx\,dy\,dz\,\triangle\Phi = k_3 + k_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Hence finally we have:

$$\iiint_G dx\,dy\,dz\,\Phi_m = -\Phi_0\, s - k_1 r_0^2 + k_2 r_0^2 + \tfrac{1}{6}(k_3 + k_4) r_0^2 \quad . \quad (24)$$

7. For *spherical particles* of radius a, for which problems of orientation do not present themselves, we have, with sufficient approximation:

$$r_0 = a; \qquad r_{max} = a; \qquad s = 4\pi a^3/3$$
$$\Phi_0 = 4/3\,a \qquad \text{from (14a)}$$
$$k_1 r_0^2 = 8\pi a^2 \quad \left.\right\} \text{neglecting terms}$$
$$k_4 = 16\pi/3 \quad \left.\right\} \text{depending on } \chi.$$

As mentioned before, in this case there are no contributions from the regions C^* and G^*, so that eq. (24), after division by Ω, ought to give the mean value required.

There remains, however, to be found the values of the quantities k_2 and k_3. Originally, in preparing the paper, I had assumed that these quantities, both of which appear to be dependent on the value of $\partial\Phi/\partial n$ at the walls of the vessel, would disappear automatically, in consequence of a mistaken application of a boundary condition. Upon reviewing the matter it appeared that in certain cases they are different from zero. It may be that when their mean value is taken for all possible positions of the particle A, this mean value will disappear, but as yet I have not succeeded in proving this. This makes it impossible at this moment to arrive at numerical results. I hope to be able to return to this point in a continuation of this paper, the more so as some remarks still must be made concerning the questions of the hydrostatic pressure and of the effective viscosity of the fluid. Further it must be considered whether the magnitude of the "viscosity effect" and the effect due to the flow of the liquid are affected when not all positions of a particle B with respect to A are equally probable, *e.g.* in consequence of repulsive forces acting between the particles. To a certain extent formulae can be derived which throw some light upon these latter problems.

(To be continued.)

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Sechste Mitteilung.) Von R. WEITZENBÖCK und W. J. BOS.

(Communicated at the meeting of October 25, 1941.)

Einleitung. Im Anschlusse an die vorhergehenden Mitteilungen untersuchen wir hier die algebraischen Regelflächen $F_2{}^2$ und $F_2{}^3$ im R_4. Im besonderen wird sich die gewöhnliche Darstellung der Regelfläche $F_2{}^3$ ergeben, wobei eine Gerade auf einen Kegelschnitt projektiv bezogen wird.

§ 19.

Bei einer Regelfläche $F_2{}^2$ zweiten Grades haben wir für die Erzeugenden:

$$\left.\begin{aligned} 0_{ik} &= p_{ik} + t\,.\,a_{ik} + \tfrac{1}{2}t^2\,.\,q_{ik} \\ 1_{ik} &= a_{ik} + t\,.\,q_{ik} \\ 2_{ik} &= q_{ik} \\ 3_{ik} &= 0 \end{aligned}\right\} \quad \ldots\ldots \quad (190)$$

und aus $M'_{00} \equiv 0\,\{t\}$ ergibt sich:

$$\left.\begin{aligned} M'_{pp} &= 0 \\ M'_{pa} &= 0 \\ M'_{pq} + M'_{aa} &= 0 \\ M'_{aq} &= 0 \\ M'_{qq} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (191)$$

Aus (190) folgt: $M'_{22} = 0$, also (Gleichung (21)) gilt $H \equiv 0$; d.h. $F_2{}^2$ liegt wenigstens in einem $R_3 : v'$. Nach (191) sind p_{ik} und q_{ik} die Koordinaten von Geraden, die nach (190) den Erzeugenden $t = 0$ und $t = \infty$ entsprechen. Man hat also:

$$v'_i = (p^2\,q^2)_i$$

und v' ist auch der Brennraum des Ebenenkomplexes a_{ik}. (191)

Die am Schluss der ersten Mitteilung gegebene Klassifikation ergibt wegen

$$M'_{02} = M'_{pq}:$$

Schneiden sich die Geraden p_{ik} und q_{ik} nicht, so ist $F_2{}^2$ eine nichtausgeartete dreidimensionale Quadrik.

Schneiden sich dagegen p_{ik} und q_{ik}, so ist $M'_{02} \equiv 0$, also $F_2{}^2$ abwickelbar. Nach (191) ist dann auch a_{ik} eine Gerade, die p_{ik} und q_{ik} trifft.

Dies gibt die Möglichkeiten:

1. a_{ik} geht nicht durch den Schnittpunkt 1 von p_{ik} und q_{ik}; dann haben wir, wenn die Punkte 1, 2 und 3 nicht derselben Geraden angehören:

$$0_{ik} = (12)_{ik} + t\,.\,(23)_{ik} + \tfrac{1}{2}t^2\,.\,(13)_{ik} \quad \text{(Kegelschnitt in 123)}$$

2. a_{ik} geht durch 1, liegt aber mit p_{ik} und q_{ik} nicht in derselben Ebene:

$$0_{ik} = (12)_{ik} + t \,.\, (14)_{ik} + \tfrac{1}{2}\, t^2 \,.\, (13)_{ik} \quad \text{(Kegel im Raum 1234)}$$

3. $a_{ik} = p_{ik} + \sigma\, q_{ik} = (12)_{ik} + \sigma\, (13)_{ik}$:

$$0_{ik} = (12)_{ik}\,(1 + t) + (13)_{ik}\,(\sigma\, t + \tfrac{1}{2}\, t^2) \quad \text{(Geradenbüschel).}$$

§ 20.

Bei einer algebraischen Regelfläche F_2^3 vom Grade 3 haben wir den Ansatz:

$$0_{ik} = p_{ik} + t \,.\, a_{ik} + \tfrac{1}{2}\, t^2 \,.\, b_{ik} + \tfrac{1}{6}\, t^3 \,.\, q_{ik} \quad . \quad . \quad . \quad . \quad (192)$$

Hieraus:

$$\left.\begin{aligned} 1_{ik} &= a_{ik} + t \,.\, b_{ik} + \tfrac{1}{2}\, t^2 \,.\, q_{ik} \\ 2_{ik} &= b_{ik} + t \,.\, q_{ik} \\ 3_{ik} &= q_{ik} \\ 4_{ik} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (193)$$

Aus $M'_{00} \equiv 0\{t\}$ folgen die Gleichungen:

$$\left.\begin{aligned} M'_{pp} = 0 \quad M'_{pa} &= 0 \\ M'_{aa} + M'_{pb} &= 0 \\ M'_{pq} + 3\, M'_{ab} &= 0 \\ 3\, M'_{bb} + 4\, M'_{aq} &= 0 \\ M'_{qb} = 0 \quad M'_{qq} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (194)$$

Damit erhalten wir:

$$M'_{02} = M'_{pb} + \tfrac{2}{3}\, t \,.\, M'_{pq} + \tfrac{1}{3}\, t^2 \,.\, M'_{aq} \,. \quad . \quad . \quad . \quad . \quad (195)$$

Wegen

$$M'_{22} = M'_{bb} + 2\, t \,.\, M'_{bq} + t^2\, M'_{qq} = -\tfrac{4}{3}\, M'_{aq} = M'_{bb}$$

erhält man für den Heftpunkt H:

$$(Hu') \equiv \Sigma\, 0_{ik}\, (u'\, M'_{22})_{ik} = \Sigma\, 0_{ik}\, (u',\, b^2\, b_1{}^2)_{ik} =$$
$$= (pu')\,(pb^2\, b_1{}^2) + t \,.\, (au')\,(ab^2\, b_1{}^2) + \tfrac{1}{2}\, t^2\, (bu')\,(bb_1{}^2\, b_2{}^2) + \tfrac{1}{6}\, t^3\, (qu')\,(qb^2\, b_1{}^2).$$

Hier gibt beim dritten Term $(bb_1{}^2 b_2{}^2)$ Null. Beim vierten wird:
$(qb^2\, b_1{}^2) = -\tfrac{4}{3}\,(qa^2\, q_1{}^2)$, also auch Null; sodass wir haben:

$$\left.\begin{aligned} (Hu') &= (pu')\,(pb^2\, b_1{}^2) + t\,(au')\,(ab^2\, b_1{}^2) = \\ &= (pu')\,(pM'_{bb}) + \tfrac{2}{3}\, t \,.\, (qu')\,(qM'_{aa}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (196)$$

Wir betrachten zuerst die allgemeine Fläche $H \not\equiv 0$, $M'_{02} \not\equiv 0$ (vgl. den Schluss der ersten Mitteilung).

Die Punkte H erfüllen in diesem Falle eine Gerade L, welche die beiden, den Parameterwerten $t = 0$ und $t = \infty$ entsprechenden Heftpunkte verbindet:

$$(1u') = (pu')\,(pM'_{bb}) = 0 \quad \text{und} \quad (2u') = (qu')\,(qM'_{aa}) \quad . \quad . \quad (197)$$

Wählen wir jetzt auf p_{ik} einen von 1 verschiedenen Punkt 3, auf q_{ik} einen von 2 verschiedenen Punkt 4, dann können wir durch diese Punkte 3 und 4 und die Erzeugende

$$\tau_{ik} = p_{ik} + \tau a_{ik} + \tfrac{1}{2}\tau^2 . b_{ik} + \tfrac{1}{6}\tau^3 q_{ik} \qquad (\tau \neq 0)$$

den R_3 mit der Gleichung

$$(x\tau^2\, 34) = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (198)$$

legen. Dieser R_3 schneidet die Erzeugende 0_{ik} von (192) in dem Punkte:

$$(u'P) = (0\tau^2\, 34)\,(0u') = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

Setzt man hier die Ausdrücke für 0_{ik} und τ_{ik} ein, so wird $(u'P)$ vom dritten Grade in t. Da aber $(u'P)$ für $t = \tau$ verschwindet, lässt sich $(u'P)$ durch $(t - \tau)$ teilen und es bleibt ein in t quadratischer Ausdruck.

Dies heisst geometrisch: der R_3 (198) schneidet die $F_2{}^3$ längs der Erzeugenden τ_{ik} und in einem Kegelschnitt, dessen Punkte durch die Erzeugenden 0_{ik} projektiv auf die der Geraden L bezogen sind.

Dies führt zu folgendem Ansatz: Wir wählen zwei Punkte 1 und 2, ihre Verbindungslinie sei die „Leitgerade" L. Weiter wählen wir einen Kegelschnitt K, in der Ebene 345 der drei Punkte 3, 4 und 5, in der Gestalt:

$$(K) \qquad z = 3 + t . 5 + t^2 . 4 \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200)$$

Hierbei soll die Determinante $(12345) \neq 0$ sein. Die Punkte K werden dann mit Hilfe des Parameters t auf die Punkte

$$(L) \qquad y = 1 + t . 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

der Leitgeraden L projektiv bezogen, sodass ein allgemeiner Punkt der $F_2{}^3$ durch die zwei Parameter t und λ dargestellt wird in der Gestalt:

$$x = y + \lambda z = 1 + t\,,\, 2 + \lambda\,(3 + t . 5 + t^2 . 4) \quad . \quad . \quad . \quad (202)$$

Bei dieser Darstellung von $F_2{}^3$ erhalten wir für die Erzeugenden $0_{ik}(t)$:

$$0_{ik} = (yz)_{ik} = (13)_{ik} + t\,[(15)_{ik} + (23)_{ik}] + t^2\,[(14)_{ik} + (25)_{ik}] + t^3\,(24)_{ik}. \quad (203)$$

Hieraus folgen:

$$1_{ik} = [(15)_{ik} + (23)_{ik}] + 2t . [(14)_{ik} + (25)_{ik}] + 3t^2 . (24)_{ik}$$
$$2_{ik} = 2\,[(14) + (25)_{ik}] + 6t . (24)_{ik}$$
$$3_{ik} = 6\,(24)_{ik}$$

und die Gleichungen (194) sind erfüllt.

Für M'_{02} und H erhalten wir jetzt:

$$(x M'_{02}) = -2\,(x\,1235) - 4t\,(x\,1234) + 2t^2\,(x\,1245) \quad . \quad . \quad (204)$$
$$(Hu') = -4\,(12345)\,[(1u') + t . (2u')] \quad . \quad . \quad . \quad . \quad (205)$$

§ 21.

Die Regelflächen $F_2{}^3$ dritter Ordnung im R_4 sind in der Litteratur ausführlich behandelt. Die verschiedenen Typen der rationalen F_2^m im R_{m+1} erhält man wie folgt: Man wählt

im R_{m+1} einen R_p und einen dazu windschiefen $R_{m-p}\,.\left(p=1,2,3\ldots\left[\frac{m}{2}\right]\right)$. Im R_p resp. R_{m-p} wählt man eine rationale Kurve C_1^p resp. C_1^{m-p}, die projektiv aufeinander bezogen werden. Die Verbindungslinien liefern dann die Erzeugenden der F_2^m im R_{m+1} [1]).

Bei $m=3$ erhalten wir $C_1{}^1=$ Leitgerade L, $C_1{}^2=$ Kegelschnitt K, und damit die im vorigen § beschriebene Konstruktion der $F_2{}^3$.

Wir führen in Kürze die wichtigsten projektiven Eigenschaften der allgemeinen $F_2{}^3$ an [2]).

$F_2{}^3$ enthält neben der Leitlinie L $(\lambda=0)$ und den Erzeugenden $0_{ik}\,(t)$ $(t=\text{const.})$ keine Geraden. Wäre nämlich M eine solche so müsste der R_3 (L,M) die ganze Fläche enthalten.

Die Gleichung der $F_2{}^3$ in Linienkordinaten bekommen wir wie folgt. Sei $x_{\lambda t}$ (202) ein allgemeiner Punkt der $F_2{}^3$, dann ist die Gleichung der Ebene $12x_{\lambda t}$:

$$\frac{1}{\lambda}\,(p^2\,12\,x_{\lambda t}) \equiv (p^2\;123)+(p^2\;125)\,t+(p^2\;124)\,t^2=0.$$

Ebenso:

$$(p^2\,13\,x_{\lambda t}) \equiv -(p^2\;123)+(p^2\;135)\,\lambda+(p^2\;134)\,\lambda\,t=0$$
$$(p^2\,24\,x_{\lambda t}) \equiv (p^2\;124)-(p^2\;234)\,\lambda+(p^2\;245)\,\lambda\,t=0$$
$$(p^2\,34\,x_{\lambda t}) \equiv (p^2\;134)+(p^2\;234)\,t+(p^2\;345)\,\lambda\,t=0$$

Wir multiplizieren die zweite und dritte dieser Gleichungen mit t, und drücken mit Hilfe der vierten Gleichung $\lambda\,t$ aus in t.

Elimination gibt dann *die Gleichung der $F_2{}^3$ in Linienkoordinaten:*

$$\begin{vmatrix} (p^2\,123) & (p^2\,125) & (p^2\,124) \\ (p^2\,135) & (p^2\,134)-(p^2\,235) & (p^2\,234) \\ (p^2\,134) & (p^2\,145)+(p^2\,234) & (p^2\,245) \end{vmatrix}=0 \quad . \quad . \quad (206)$$

Diese Gleichung ist also vom dritten Grade in p_{ik}, wie auch geometrisch zu erwarten war. Denn durch einen Punkt gehen im allgemeinen 3 Treffgeraden der $F_2{}^3$ die in einer Ebene liegen.

Eine Ebene w^3 schneidet die $F_2{}^3$ nämlich im allgemeinen in drei Punkten. Die zugehörigen drei Erzeugenden ergeben sich nach (203) aus der Gleichung:

$$(w^3\,12)+t\,.\,[(w^3\,15)+(w^3\,23)]+t^2\,.\,[(w^3\,25)+(w^3\,14)]+t^3\,.\,(w^3\,24)=0. \quad (207)$$

Die Diskriminante $R(w_{ijk})$ dieser kubischen Form wird vom Grade vier in den Ebenenkoordinaten w_{ijk}; $R(w_{ijk})=0$ kann also als *Gleichung der Fläche in Ebenenkoordinaten* angesehen werden.

Geometrisch kann man sagen: Ein Raum R_3 durch eine Gerade N schneidet die $F_2{}^3$ im allgemeinen in einer rationalen Kurve $C_1{}^3$. (Siehe unten, Gl. (210). Die Tangentenfläche dieser Kurve ist von vierter Ordnung. Es gibt also im Raume R_3 vier Tangenten der

---

[1]) C. SEGRE, Atti Torino **19**, 355 (1884); und Math. Ann. **30**, 203—226 (1887); **34**, 1—25 (1889); W. L. EDGE, The theory of ruled surfaces. Cambridge. 1931, section III; E. BERTINI, Introduzione alla geometria proiettiva degli iperspazi, Pisa, 285—327 (1907).

[2]) Vgl. neben den genannten Arbeiten von Segre und Edge (S. 36—42) auch: G. VERONESE, Math. Ann. **19**, 229—232 (1882); und besonders O. BOTTEMA, Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**, 1282—1287 (1940).

Kurve C_1^3 die N schneiden, und also auch vier Tangentenebenen der Fläche F_2^3 durch N in einem R_3.

Die Schnittfigur von w^3 mit der Fläche kann sein: drei getrennte nicht auf einer Geraden liegende Punkte, ein doppelt zu zählender und ein einfacher Punkt, ein dreifach zu zählender Punkt, eine Gerade (L), eine Erzeugende und ein Punkt, zwei sich schneidende Geraden (L und eine Erzeugende), ein Kegelschnitt.

Auf der F_2^3 liegen ∞^2 Kegelschnitte K, die in Flächenparametern (t, λ) durch

$$\frac{1}{\lambda} = at + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

definiert werden. Sind a und b gegeben, so werden die Koordinaten der Punkte des zugehörigen Kegelschnittes nach (202):

$$\varrho\, x = (1 + t \,.\, 2)\,(a \,.\, t + b) + (3 + t \,.\, 5 + t^2 \,.\, 4) =$$
$$= (b \,.\, 1 + 3) + t \,.\, (a \,.\, 1 + b \,.\, 2 + 5) + t^2 \,.\, (a \,.\, 2 + 4).$$

Hieraus ergibt sich leicht: K schneidet jede Erzeugende in einem und nur einem Punkte; K kann bestimmt werden durch zwei Punkte der F_2^3, die nicht auf der Leitlinie und nicht auf denselben Erzeugenden liegen. Zwei Kegelschnitte $K_1(a_1 b_1)$ und $K_2(a_2 b_2)$ haben nur einen Punkt gemeinsam. Durch einen gegebenen Punkt (t_0, λ_0) auf F_2^3 geht ein Büschel von Kegelschnitten K, das durch

$$\varrho\, x = a\,(t - t_0)\,(1 + t \,.\, 2) + \left[\frac{1}{\lambda_0}\,(1 + t \,.\, 2) + (3 + t \,.\, 5 - t^2 \,.\, 4)\right]$$
$$= aK_1 + K_2$$

dargestellt werden kann. Dabei zerfällt also K_1 in die Erzeugende $t = t_0$ und die Leitlinie L.

Ein R_3 (v') schneidet die F_2^3 in einer rationalen C_1^3 die in Flächenparametern durch

$$\lambda = -\frac{v_1' + v_2' \,.\, t}{v_3' + v_5' \,.\, t + v_4' \,.\, t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

festgehalten werden kann.

C_1^3 kann zerfallen: in eine Erzeugende und einen Kegelschnitt, in zwei getrennte Erzeugende und die Leitlinie L, und schliesslich in eine doppelt zu zählende Erzeugende und L.

Die C_1^3 ist gegeben durch:

$$\varrho\, x = (1 + t \,.\, 2)\,(v_3' + t \,.\, v_5' + t^2 \,.\, v_4') - (3 + t \,.\, 5 + t^2 \,.\, 4)\,(v_1' + t \,.\, v_2'). \quad (210)$$

Die rationale Kurve C_1^3 kann nur einen Doppelpunkt haben wenn sie zerfällt und also wenigstens eine Erzeugende enthält.

Die Gleichung aller Räume u', in welchen eine Erzeugende $0_{ik}(t)$ der F_2^3 liegt, ist also auch die Gleichung der F_2^3 in Raumkoordinaten.

Der R_3 u' enthält $0_{ik}(t)$ wenn:

$$0_{u'}\, 0_{v'} \equiv 0\,\{v'\}.$$

Mit einem Hilfpunkte y und (203) bekommen wir die Eliminationsgleichung:

$$(1_{u'}\, 3_{v'} - 3_{u'}\, 1_{v'},\ 1_{u'}\, 5_{v'} - 5_{u'}\, 1_{v'} + 2_{u'}\, 3_{v'} - 3_{u'}\, 2_{v'},\ 1_{u'}\, 4_{v'} - 4_{u'}\, 1_{v'} +$$
$$+ 2_{u'}\, 5_{v'} - 5_{u'}\, 2_{v'},\ 2_{u'}\, 4_{v'} - 4_{u'}\, 2_{v'},\ y_{v'}) \equiv 0.$$

Dies gibt 15 Terme die wir so zusammenfassen

$$(1_{u'}\,1_{u'}\,4_{u'} - 1_{u'}\,2_{u'}\,5_{u'} + 2_{u'}\,2_{u'}\,3_{u'})\,[(2345\,y)\,1_{u'} - (1345\,y)\,2_{u'} + \\ + (1245\,y)\,3_{u'} - (1235\,y)\,4_{u'} + (1234\,y)\,5_{u'}]$$

und die zwei Terme

$$(1234\,y)\,3_{u'}\,2_{u'}\,1_{u'}\,4_{u'} - (1234\,y)\,1_{u'}\,3_{u'}\,4_{u'}\,2_{u'}$$

die Null ergeben.

Hieraus erhält man *die Gleichung der F_2^3 in Raumkoordinaten u'*:

$$(12345)\,(1_{u'}\,1_{u'}\,4_{u'} - 1_{u'}\,2_{u'}\,5_{u'} + 2_{u'}\,2_{u'}\,3_{u'}) = 0 \quad . \quad . \quad (211a)$$

Die Gleichung ist also vom dritten Grade in u'. Geometrisch: Eine allgemeine Ebene schneidet die F_2^3 in drei Punkten; jedes Paar dieser Punkte bestimmt einen Kegelschnitt auf F_2^3 und also einen Raum R_3.

Durch eine Ebene gibt es infolgedessen drei Räume, welche die F_2^3 in einer ausgearteten Kurve C_1^3 schneiden.

Mit Hilfe der Gleichung (203) rechnet man leicht nach, dass auch die Differentialkontravariante

$$(0^2\,1\,2\,3)\,1_{u'}\,2_{u'}\,3_{u'} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (211b)$$

die F_2^3 in Raumkoordinaten darstellt.

Wir bemerken weiter noch:

Die F_2^3 ist durch 18 Konstanten bestimmt und gestattet ∞^6 projektive Transformationen in sich. Geometrisch kann die F_2^3 gegeben werden durch 4 Geraden, die eine gemeinsame Transversale besitzen.

Mathematics. — *Une propriété des variétés de* VERONESE. Par LUCIEN GODEAUX. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 27, 1941.)

Appelons variété de VERONESE à n dimensions la variété obtenue en rapportant projectivement les hyperquadriques d'un espace linéaire à n dimensions aux hyperplans d'un espace linéaire à $\frac{1}{2}n(n+3)$ dimensions. Pour $n=2$, on obtient la surface de VERONESE. On peut obtenir les propriétés d'une variété de VERONESE en généralisant la méthode suivie dans le cas d'une surface [1]). Dans cette note, nous nous proposons d'obtenir quelques propriétés de cette variété par une autre voie. Nous utilisons la variété de SEGRE représentant les couples de points de deux espaces linéaires à n dimensions [2]); cette variété contient une involution du second ordre représentée par la variété des cordes de la variété de VERONESE à n dimensions. C'est sur cette remarque qu'est basée la méthode suivie.

1. Soient S_n un espace linéaire à n dimensions, y, z deux de ses points, $y_0, y_1, \ldots, y_n$ et $z_0, z_1, \ldots, z_n$ leurs coordonnées projectives homogènes. Désignons par V_{2n} la variété de SEGRE représentant les couples ordonnés y, z et par Ω_{2n} la variété représentant les couples non ordonnés y, z.

Nous prendrons comme modèle projectif de la variété V_{2n} la variété d'ordre $(2n)!:(n!)^2=\binom{2n}{n}$ appartenant à l'espace linéaire S_r à $r=n(n+2)$ dimensions, obtenue en posant

$$\varrho\, X_{ik}=y_i\, z_k. \quad (i,k=0,1,2,\ldots,n)$$

et en interprétant les X_{ik} comme les coordonnées projectives d'un point X de cet espace.

Pour modèle projectif de la variété Ω_{2n}, nous prendrons la variété appartenant à l'espace S_ϱ à $\varrho=\frac{1}{2}n(n+3)$ dimensions, obtenue en posant

$$\varrho\, Y_{ik}=y_i\, z_k+y_k\, z_i$$

et en prenant les Y_{ik} comme coordonnées projectives d'un point Y de cet espace.

Les équations de la variété V_{2n} s'obtiennent en écrivant que le déterminant

$$|X_{ik}|$$

est de caractéristique un; celles de la variété Ω_{2n} en écrivant que le déterminant *symétrique*

$$|Y_{ik}|$$

est de caractéristique deux.

Considérons, dans l'espace S_r, l'homographie H d'équations

$$\varrho\, X'_{ik}=X_{ki}.$$

Cette homographie est harmonique et ses axes ponctuels sont

a) Un espace linéaire σ à ϱ dimensions, représenté par les équations

$$X_{ik}=X_{ki}.$$

---

[1]) Voir par exemple: E. BERTINI, *Introduzione alla Geometria proiettiva degli iperspazi* (Pisa, Spoerri, 1907).

[2]) C. SEGRE, *Sulle varietà che rappresentano le coppie di punti di due piani o spazi* (Rendiconti del Circolo Matematico di Palermo, 1891, pp. 192—204).

Cet espace coupe V_{2n} suivant une variété U dont les équations s'obtiennent en écrivant que le déterminant *symétrique* $|X_{ik}|$ est de caractéristique un. Cette variété est obtenue en rapportant projectivement aux hyperplans de σ les hyperquadriques d'un espace linéaire à n dimensions. Pour cette raison, nous l'appellerons la variété de VERONESE à n dimensions. L'ordre de la variété U est égal à 2^n.

b) Un espace linéaire σ' à $\frac{1}{2}(n-1)(n+2)$ dimensions, d'équations

$$X_{ik} + X_{ki} = 0.$$

Cet axe ne rencontre pas la variété V_{2n}.

La variété V_{2n} est transformée en elle-même par l'homographie H et celle-ci engendre donc, sur cette variété, une involution I_2, d'ordre deux, dont les points unis forment la variété de VERONESE U.

2. Les hyperplans ξ passant par l'axe σ' de H sont unis pour cette homographie et découpent, sur V_{2n}, un système linéaire $|F|$, de variétés à $2n-1$ dimensions, de dimension ϱ, appartenant à l'involution I_2. Pour obtenir une variété image de cette involution, rapportons projectivement les variétés F aux hyperplans de l'espace S_ϱ, à ϱ dimensions.

L'équation des hyperplans ξ étant

$$\Sigma \lambda_{ik}(X_{ik} + X_{ki}) = 0,$$

il suffit de poser

$$\varrho\, Y_{ik} = X_{ik} + X_{ki} = y_i z_k + y_k z_i.$$

On obtient donc ainsi les équations de la variété Ω_{2n}. On voit en effet immédiatement que si, dans un déterminant tiré de $|Y_{ik}|$ en effaçant $n-2$ lignes et $n-2$ colonnes, on remplace Y_{ik} par $X_{ik} + X_{ki}$, ce déterminant se ramène à une somme de déterminants identiquement nuls en raison des équations de V_{2n}.

D'ailleurs, on passe des équations de V_{2n} à celles de Ω_{2n} en considérant l'identité $\varrho\, y_i = z_i$, qui fait correspondre au couple ordonné $y\,z$, le couple ordonné $z\,y$. L'identité donne d'ailleurs les équations de l'homographie H et inversement, on déduit des équations de cette homographie $\varrho\, y_i = z_i$.

Les variétés Φ, qui correspondent sur Ω_{2n} aux variétés F, sont les sections hyperplanes de Ω_{2n}; elles forment un système de degré $\frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n-1}$.

A la variété U de V_{2n}, caractérisée par $X_{ik} = X_{ki}$, correspond sur Ω_{2n} la variété de diramation D, dont les équations s'obtiennent en écrivant que le déterminant $|Y_{ik}|$ est de caractéristique un. La variété D est donc une variété de VERONESE à n dimensions; elle est double pour la variété Ω_{2n}.

3. Envisageons maintenant les hyperplans ξ' de S_r passant par l'axe σ de l'homographie H; ils ont pour équation

$$\Sigma \lambda_{ik}(X_{ik} - X_{ki}) = 0, \quad (i < k)$$

et découpent sur V_{2n} des variétés F' à $2n-1$ dimensions, formant un système linéaire de dimension $\frac{1}{2}(n-1)(n+2)$, appartenant à l'involution I_2. Désignons par Φ' les variétés qui correspondent sur Ω_{2n} aux variétés F'.

Pour obtenir l'équation des variétés Φ', élevons les deux membres de l'équation précédente au carré; nous obtenons

$$\Sigma \lambda_{ik}\lambda_{jh}(X_{ik}X_{jh} + X_{ki}X_{hj} - X_{ik}X_{hj} - X_{ki}X_{jh}) = 0,$$

ce qui s'écrit

$$\Sigma \lambda_{ik}\lambda_{jh} \begin{vmatrix} Y_{jk} & Y_{ji} \\ Y_{hk} & Y_{hi} \end{vmatrix} = 0.$$

Cette équation représente une hyperquadrique de S_ϱ, passant par la variété de VERONESE D. Lorsque les coefficients λ varient, cette hyperquadrique engendre un système continu Σ, de dimension $\frac{1}{2}(n-1)(n+2)$ et d'indice $2^{\frac{1}{2}(n-1)(n+2)}$. Ces hyperquadriques découpent sur Ω_{2n} les variétés Φ'.

Soit $\overline{F}$ une section hyperplane quelconque de V_{2n}. A cette variété correspond sur Ω_{2n} une variété $\overline{\Phi}$ qui correspond également à la section hyperplane que H fait correspondre à F. Lorsque $\overline{F}$ varie sur V_{2n}, $\overline{\Phi}$ engendre un système continu rationnel, dont les éléments sont par conséquent contenus totalement dans un système linéaire $|\overline{\Phi}|$. Aux variétés de ce système correspondent sur V_{2n} des variétés appartenant au système $|2\overline{F}|$.

Faisons varier $\overline{F}$ d'une manière continue dans le système $|\overline{F}|$ des sections hyperplanes de V_{2n} en la faisant tendre vers une variété F découpée par un hyperplan passant par l'axe σ' de H. La variété $\overline{\Phi}$ varie d'une manière continue dans $|\overline{\Phi}|$ et tend vers une section hyperplane Φ de Ω_{2n}, comptée deux fois. On a donc

$$\overline{\Phi} | = | 2\,\Phi \ .$$

Faisons maintenant varier F d'une manière continue en la faisant tendre vers une variété F' découpée par un hyperplan passant par l'axe σ de H. La variété correspondante $\overline{\Phi}$ tend vers une variété Φ' augmentée de la variété à $2n-1$ dimensions formée par les points infiniment voisins de D, situés sur Ω_{2n}. On a donc

$$|\overline{\Phi}| = |2\,\Phi' + D| = |2\,\Phi|\,.$$

Il en résulte que les variétés $\overline{\Phi}$ sont découpées sur Ω_{2n} par les hyperquadriques de S_ϱ et que le long de chaque variété Φ', il y a une hyperquadrique de Σ touchant Ω_{2n} en chaque point d'intersection. Les variétés Φ' sont par suite d'ordre $\frac{1}{2}\binom{2n}{n}$.

La variété Ω_{2n} est l'enveloppe du système Σ, ayant pour base la variété de VERONESE *D.*

4. A chaque point Y de S_ϱ, faisons correspondre dans S_n l'hyperquadrique enveloppe d'équation

$$\Sigma\, \lambda_{ik}\, \xi_i\, \xi_k = 0\,.$$

Cette correspondance est projective.

Aux points de Ω_{2n} correspondent les hyperquadriques $n-1$ fois spécialisées, c'est-à-dire dégénérées en deux gerbes d'hyperplans. Aux points de D correspondent les hyperquadriques n fois spécialisées, c'est-à-dire dégénérées en deux gerbes d'hyperplans superposées.

Considérons deux points Y', Y'' de la variété D et soient y', y'' les centres des gerbes d'hyperplans correspondantes dans S_n. Aux points de la droite $Y'\,Y''$ correspondent dans S_n les hyperquadriques

$$(\Sigma\, y'_i\, \xi_i)^2 + \lambda\,(\Sigma\, y''_i\, \xi_i)^2 = 0\,.$$

Toutes les hyperquadriques de ce système sont $n-1$ fois spécialisées; chacune d'elles dégénère en deux gerbes d'hyperplans dont les sommets forment un couple d'une involution appartenant à la droite $y'\,y''$. Les points doubles de cette involution correspondent aux deux points d'appui Y', Y'' de la droite $Y'\,Y''$ sur la variété D.

En particulier, lorsque le point Y'' tend vers Y' sur D, c'est-à-dire lorsqu'il s'agit d'une droite tangente en Y' à la variété D, aux points de cette droite correspondent dans S_n les couples de points d'une involution parabolique sur une droite, c'est-à-dire les couples de points d'une droite dont l'un, y', est fixe.

Les cordes de la variété D sont, d'après ceci, situées dans la variété Ω_{2n}.

5. La variété V_{2n} contient deux séries ∞^n d'espaces à n dimensions. Un espace η de la première série est le lieu des points de la variété qui représentent les couples de points de S_n dont le premier, y, est fixe. Un espace ζ de la seconde série représente les couples de points dont le second, z, est fixe. Deux espaces de la même série ne se rencontrent pas; un espace η et un espace ζ ont en commun un point.

A un espace η, l'homographie H fait correspondre un espace ζ et réciproquement. Le point commun à ces deux espaces se trouve sur la variété unie U.

Les équations de l'espace η associé au point y sont

$$\frac{X_{0i}}{y_0} = \frac{X_{1i}}{y_1} = \ldots = \frac{X_{ki}}{y_k} = \ldots = \frac{X_{ni}}{y_n}$$

et celles de l'espace ζ associé au point z,

$$\frac{X_{i0}}{z_0} = \frac{X_{i1}}{z_1} = \ldots = \frac{X_{ik}}{z_k} = \ldots = \frac{X_{in}}{z_n}.$$

Si ces deux espaces sont conjugués par rapport à H, on a $y = z$ et à ces deux espaces, correspond, sur la variété Ω_{2n}, l'espace à n dimensions d'équations

$$y_i^2 Y_{kk} - 2 y_i y_k Y_{ki} + y_k^2 Y_{ii} = 0.$$

On reconnaît immédiatement que les droites passant par le point Y et appartenant à cet espace, sont tangentes à D au point Y. Il en résulte que

Aux espaces à n dimensions η, ζ de V_{2n} conjugés dans l'homographie H, correspondent dans Ω_{2n} les espaces à n dimensions tangents à la variété de VERONESE *D.*

6. Considérons la variété de SEGRE V' à $2(n-1)$ dimensions, représentant sur V_{2n} les couples de point y, z de deux hyperplans

$$\mu_0 y_0 + \mu_1 y_1 + \ldots + \mu_n y_n = 0,$$
$$\nu_0 z_0 + \nu_1 z_1 + \ldots + \nu_n z_n = 0$$

de S_n. Cette variété V' est découpée sur V_{2n} par l'espace linéaire à $(n-1)(n+1)$ dimensions

$$\mu_0 X_{0i} + \mu_1 X_{1i} + \ldots + \mu_n X_{ni} = 0,$$
$$\nu_0 X_{i0} + \nu_1 X_{i1} + \ldots + \nu_n X_{in} = 0.$$

Pour que la variété V' soit transformée en elle-même par l'homographie H, il faut et il suffit que l'on ait

$$\frac{\mu_0}{\nu_0} = \frac{\mu_1}{\nu_1} = \ldots = \frac{\mu_n}{\nu_n}.$$

A cette variété correspond alors sur Ω_{2n} une variété Ω', située dans l'espace à $\frac{1}{2}(n-1)(n+2)$ dimensions

$$\mu_0 Y_{0i} + \mu_1 Y_{1i} + \ldots + \mu_n Y_{ni} = 0.$$

Considérons la section de la variété de VERONESE D par cet espace. Il lui correspond dans S_n l'hyperplan

$$\mu_0 x_0 + \mu_1 x_1 + \ldots + \mu_n x_n = 0.$$

Par conséquent, la section considérée est une variété de VERONESE D' à $n-1$ dimensions. Cette variété détermine complètement l'espace linéaire à $\frac{1}{2}(n-1)(n+2)$ dimensions. Les cordes de D' étant des cordes de D, appartiennent à Ω_{2n}. L'espace linéaire déterminé par D' coupe donc Ω_{2n} suivant une variété de même construction à $2(n-1)$ dimensions.

Liège, le 28 juin 1941.

Mathematics. — *Multiplikationstheoreme für die Funktion* $G_{p,q}^{m,n}(z)$. Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 25, 1941.)

§ 1. Einleitung.

Für die BESSELschen Funktionen $J_\nu(z)$, $Y_\nu(z)$ und $K_\nu(z)$ gelten bekanntlich die sogenannten Multiplikationstheoreme [1])

$$\lambda^{-\nu} J_\nu(\lambda z) = \sum_{r=0}^{\infty} \frac{(1-\lambda^2)^r (\frac{1}{2} z)^r}{r!} J_{\nu+r}(z), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\lambda^{\nu} J_\nu(\lambda z) = \sum_{r=0}^{\infty} \frac{(\lambda^2-1)^r (\frac{1}{2} z)^r}{r!} J_{\nu-r}(z), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\lambda^{-\nu} Y_\nu(\lambda z) = \sum_{r=0}^{\infty} \frac{(1-\lambda^2)^r (\frac{1}{2} z)^r}{r!} Y_{\nu+r}(z), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\lambda^{\nu} Y_\nu(\lambda z) = \sum_{r=0}^{\infty} \frac{(\lambda^2-1)^r (\frac{1}{2} z)^r}{r!} Y_{\nu-r}(z) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

und

$$\lambda^{-\nu} K_\nu(\lambda z) = \sum_{r=0}^{\infty} \frac{(1-\lambda^2)^r (\frac{1}{2} z)^r}{r!} K_{\nu+r}(z). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Verwandte Formeln für die WHITTAKERschen Funktionen $M_{k,m}(z)$ und $W_{k,m}(z)$ sind von Hernn ARTUR ERDÉLYI gefunden worden; ERDÉLYI beweist nämlich [2])

$$\lambda^{-m-\frac{1}{2}} e^{-\frac{1}{2}\lambda z} M_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(1+2m)\,\Gamma(\frac{1}{2}+k+m+r)\,(1-\lambda)^r}{r!\,\Gamma(1+2m+r)\,\Gamma(\frac{1}{2}+k+m)} z^{\frac{1}{2}r} e^{-\frac{1}{2}z} M_{k+\frac{1}{2}r,\,m+\frac{1}{2}r}(z), \quad (6)$$

$$\lambda^k e^{\frac{1}{2}\lambda z} M_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k+m+r)}{r!\,\Gamma(\frac{1}{2}+k+m)} \left(1-\frac{1}{\lambda}\right)^r e^{\frac{1}{2}z} M_{k+r,m}(z) \quad (7)$$

und

$$\lambda^k e^{-\frac{1}{2}\lambda z} W_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\lambda}-1\right)^r e^{-\frac{1}{2}z} W_{k+r,m}(z). \quad . \quad . \quad (8)$$

In der vorliegenden Note werde ich zeigen, dass die Beziehungen (1), . . . , (8) nur Spezialfälle viel allgemeinerer Formeln sind. Ich werde nämlich Multiplikationstheoreme ableiten für die Funktion $G_{p,q}^{m,n}(z)$, die ich auch in einigen anderen Arbeiten [3]) betrachtet habe, und ich werde zeigen, dass nicht nur die obengegebenen Multiplikationsformeln der BESSELschen und WHITTAKERschen Funktionen, sondern auch verschiedene analog gebauten Relationen zum Vorschein kommen durch Spezialisierung der Multiplikationsformeln der G-Funktion.

[1]) Man vergl. WATSON, [**10**], 140—142.

(1) gilt für alle Werte von λ; in (2), (3), (4) und (5) wird $|\lambda^2 - 1| < 1$ vorausgesetzt.

[2]) [**3**], Formeln (6, 1), (5, 1) und (5, 5); [**4**], Formeln (6, 3) und (6, 1). Siehe auch DHAR, [**2**], Formel (47) und GOLDSTEIN, [**5**], Formel (46).

In (6) ist λ beliebig; (7) und (8) gelten für $\Re(\lambda) > \frac{1}{2}$.

[3]) [**6**], [**7**], [**8**] und [**9**].

§ 2. Die Multiplikationstheoreme der G-Funktion.

Die Funktion $G_{p,q}^{m,n}(z)$ wird definiert durch

$$G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right)=\sum_{h=1}^{m}\frac{\prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j-b_h)\prod_{j=1}^{n}\Gamma(1+b_h-a_j)}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_h)}z^{b_h}\times\Bigg\{ \tag{9}$$

$$\times {}_pF_{q-1}(1+b_h-a_1,\ldots,1+b_h-a_p;1+b_h-b_1,..^{*}..,1+b_h-b_q;(-1)^{m+n+p}z)\Bigg\}.$$

Hierin wird $z\neq 0$,

$$0\leqq n\leqq p<q,\qquad 1\leqq m\leqq q, \tag{10}$$

b_j-b_h nicht ganz $(j=1,\ldots,m;\ h=1,\ldots,m;\ j\neq h)$

und

$$a_j-b_h\neq 1,2,3,\ldots\ (j=1,\ldots,n;\ h=1,\ldots,m)$$

vorausgesetzt; der Stern bedeutet, dass die Zahl $1+b_h-b_h$ im System $1+b_h-b_1,\ldots,1+b_h-b_q$ gestrichen werden muss.

Die Funktion $G_{p,q}^{m,n}(z)$ ist eine symmetrische Funktion von $b_1,\ldots,b_m$, von $b_{m+1},\ldots,b_q$, von $a_1,\ldots,a_n$ und von $a_{n+1},\ldots,a_p$. Die Parameter $b_1,\ldots,b_q$ und $a_1,\ldots,a_p$ können also in vier Klassen verteilt werden [4]). Jeder Klasse entspricht ein Multiplikationstheorem.

Erstes Multiplikationstheorem. *Ist* $|\lambda-1|<1$, *so gilt*

$$\lambda^{-b_1}G_{p,q}^{m,n}\left(\lambda w\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right)=\sum_{r=0}^{\infty}\frac{(1-\lambda)^r}{r!}G_{p,q}^{m,n}\left(w\left|\begin{matrix}a_1,\ldots,a_p\\ r+b_1,b_2,\ldots,b_q\end{matrix}\right.\right). \tag{11}$$

Ist $m=1$, *so gilt* (11) *für alle Werte von* λ.

Zweites Multiplikationstheorem. *Ist* $m<q$ *und* $|\lambda-1|<1$, *so gilt*

$$\lambda^{-b_q}G_{p,q}^{m,n}\left(\lambda w\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right)=\sum_{r=0}^{\infty}\frac{(\lambda-1)^r}{r!}G_{p,q}^{m,n}\left(w\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_{q-1},r+b_q\end{matrix}\right.\right). \tag{12}$$

Drittes Multiplikationstheorem. *Ist* $n\geqq 1$ *und* $\Re(\lambda)>\frac{1}{2}$, *so gilt*

$$\lambda^{1-a_1}G_{p,q}^{m,n}\left(\lambda w\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right)=\sum_{r=0}^{\infty}\frac{1}{r!}\left(1-\frac{1}{\lambda}\right)^r G_{p,q}^{m,n}\left(w\left|\begin{matrix}-r+a_1,a_2,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right). \tag{13}$$

Viertes Multiplikationstheorem. *Ist* $n<p$ *und* $\Re(\lambda)>\frac{1}{2}$, *so gilt*

$$\lambda^{1-a_p}G_{p,q}^{m,n}\left(\lambda w\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right)=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{1}{\lambda}-1\right)^r G_{p,q}^{m,n}\left(w\left|\begin{matrix}a_1,\ldots,a_{p-1},-r+a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right). \tag{14}$$

[4]) Man vergl. auch (21).

§ 3. Hilfsformeln.

Ich werde die folgenden Relationen beweisen:

$$\frac{d}{dz}\left\{z^{-b_1}G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)\right\}=-z^{-1-b_1}G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ 1+b_1,b_2,\ldots,b_q\end{matrix}\right), \quad (15)$$

$$\frac{d}{dz}\left\{z^{-b_q}G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)\right\}=z^{-1-b_q}G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_{q-1},1+b_q\end{matrix}\right) \quad (m<q), \quad (16)$$

$$\frac{d}{dz}\left\{z^{\,1+a_1}G^{m,n}_{p,q}\left(\frac{1}{z}\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)\right\}=-z^{\,2+a_1}G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}1-1+a_1,a_2,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) \quad (n\geqq 1), \quad (17)$$

$$\frac{d}{dz}\left\{z^{-1+a_p}G^{m,n}_{p,q}\left(\frac{1}{z}\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)\right\}=z^{-2+a_p}G^{m,n}_{p,q}\left(\frac{1}{z}\,\middle|\,\begin{matrix}a_1,\ldots,a_{p-1},-1+a_p\\ b_1,\ldots,b_q\end{matrix}\right) \quad (n<p). \quad (18)$$

Beim Beweis nehme ich vorläufig an, dass

$$m+n-\tfrac{1}{2}p-\tfrac{1}{2}q>0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

und

$$|\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

ist. Sind diese zwei Bedingungen erfüllt, so besitzt die Funktion $G^{m,n}_{p,q}(z)$ die Integraldarstellung [5])

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s\,ds; \quad (21)$$

hierin läuft der Integrationsweg L von $-\infty i+\tau$ nach $\infty i+\tau$ (τ ist eine beliebige reelle Zahl) und zwar so, dass die Punkte

$$b_j,\ b_j+1,\ b_j+2,\ldots\ (j=1,\ldots,m)$$

auf der rechten, die Punkte

$$a_j-1,\ a_j-2,\ a_j-3,\ldots\ (j=1,\ldots,n)$$

hingegen auf der linken Seite von L liegen.

Aus (21) folgt

$$\frac{d}{dz}\left\{z^{-b_1}G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)\right\}$$

$$=-\frac{z^{-1-b_1}}{2\pi i}\int_L \frac{\Gamma(1+b_1-s)\prod_{j=2}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s\,ds,$$

und diese Beziehung ist wegen (21) gleichbedeutend mit (15); Formel

[5]) Man vergl. BARNES, [1], 65–71.

(15) ist also bewiesen für den Fall, dass die Bedingungen (19) und (20) erfüllt sind.

Unter Annahme der Voraussetzungen (19) und (20) liefert man ganz analog den Beweis von (16), (17) und (18).

Es ist leicht einzusehen, auf Grund der Theorie der analytischen Fortsetzung, dass Voraussetzung (20) bei (15), (16), (17) und (18) ausfallen kann. Ich werde noch zeigen, dass auch Voraussetzung (19) fortgelassen werden darf.

Ich betrachte also den Fall mit $m+n-\frac{1}{2}p-\frac{1}{2}q \leqq 0$. Wegen (10) gilt dann $m+n-q<0$, somit $m+1 \leqq q$, so dass die Funktion $G_{p,q}^{m+1,n}(z)$ einen Sinn hat (man vergl. (10)). Die Reduktionsformel [6])

$$G_{p,q}^{m,n}\left(z \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) = \frac{1}{2\pi i}\left\{ e^{\pi i\, b_{m+1}}\, G_{p,q}^{m+1,n}\left(z\, e^{-\pi i} \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) \right. \tag{22}$$
$$\left. - e^{-\pi i\, b_{m+1}}\, G_{p,q}^{m+1,n}\left(z\, e^{\pi i} \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right)\right\}$$

setzt uns nun in stand bei (15), (17) und (18) den Fall mit $m+n-\frac{1}{2}p-\frac{1}{2}q \leqq 0$ aus dem Fall mit $m+n-\frac{1}{2}p-\frac{1}{2}q>0$ abzuleiten.

Was (16) anbetrifft, ich werde zunächst zeigen, dass (16) gewiss gilt, falls $m=q-1$ ist. Da $G_{p,q}^{q,n}(z)$ eine symmetrische Funktion von $b_1,\ldots,b_q$ ist, folgt nämlich aus (15)

$$\frac{d}{dz}\left\{ z^{-b_q}\, G_{p,q}^{q,n}\left(z \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right)\right\} = -z^{-1-b_q}\, G_{p,q}^{q,n}\left(z \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{q-1},1+b_q \end{matrix}\right.\right). \tag{23}$$

Nun hat man wegen (22), mit $m=q-1$ angewendet,

$$z^{-b_q}\, G_{p,q}^{q-1,n}\left(z \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right) = \frac{1}{2\pi i}\left\{ (z\, e^{-\pi i})^{-b_q}\, G_{p,q}^{q,n}\left(z\, e^{-\pi i} \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right)\right.$$
$$\left. - (z\, e^{\pi i})^{-b_q}\, G_{p,q}^{q,n}\left(z\, e^{\pi i} \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right)\right\}.$$

Hieraus und aus (23) geht hervor

$$\frac{d}{dz}\left\{ z^{-b_q}\, G_{p,q}^{q-1,n}\left(z \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right.\right)\right\} = \frac{1}{2\pi i}\left\{ (z\, e^{-\pi i})^{-1-b_q}\, G_{p,q}^{q,n}\left(z\, e^{-\pi i} \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{q-1},1+b_q \end{matrix}\right.\right)\right.$$
$$\left. - (z\, e^{\pi i})^{-1-b_q}\, G_{p,q}^{q,n}\left(z\, e^{\pi i} \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{q-1},1+b_q \end{matrix}\right.\right)\right\} = z^{-1-b_q}\, G_{p,q}^{q-1,n}\left(z \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_{q-1},1+b_q \end{matrix}\right.\right)$$

wegen (22), mit $m=q-1$ und $1+b_q$ statt b_q angewendet.

Formel (16) ist also bewiesen, falls $m=q-1$ ist. Der Fall mit $m<q-1$ kann wieder mit Hilfe von (22) aus dem Fall mit $m=q-1$ abgeleitet werden.

---

6) Beziehung (22) kann leicht aus (9) abgeleitet werden.

Zum Schluss gebe ich noch eine Hilfsformel, die ich in § 5 wiederholt benutzen werde:

$$G_{p,q}^{m,n}\left(\zeta \,\middle|\, \begin{matrix} a_1+\tau,\ldots,a_p+\tau \\ b_1+\tau,\ldots,b_q+\tau \end{matrix}\right) = \zeta^{\tau}\, G_{p,q}^{m,n}\left(\zeta \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right). \quad . \quad (24)$$

§ 4. Beweis der Multiplikationstheoreme.

Die Funktion

$$z^{-b_1}\, G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

ist für alle Werte von z, $z=0$ ausgenommen, eine analytische Funktion von z. Auf Grund des TAYLORschen Satzes hat man also, falls $|\eta| < |w|$ ist,

$$(w+\eta)^{-b_1}\, G_{p,q}^{m,n}\left(w+\eta \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \sum_{r=0}^{\infty} \frac{\eta^r}{r!} \frac{d^r}{(dw)^r} \left\{ w^{-b_1}\, G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)\right\}. \quad (26)$$

Nun folgt aus (15), dass

$$\frac{d^r}{(dw)^r}\left\{ w^{-b_1}\, G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)\right\} = (-1)^r\, w^{-r-b_1}\, G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ r+b_1, b_2,\ldots,b_q \end{matrix}\right)$$

ist. Für $|\eta| < |w|$ gilt daher die TAYLORsche Reihenentwicklung

$$(w+\eta)^{-b_1}\, G_{p,q}^{m,n}\left(w+\eta \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \sum_{r=0}^{\infty} \frac{(-1)^r \eta^r w^{-r-b_1}}{r!}\, G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ r+b_1, b_2,\ldots,b_q \end{matrix}\right). \quad (27)$$

ersetzt man hierin η durch $\lambda w - w$, so bekommt man (11).

Ist $m=1$, so ist die Funktion (25) für jedes z, $z=0$ *nicht* ausgenommen, eine analytische Funktion von z; Beziehung (27) ist dann gültig für alle Werte von η, Entwicklung (11) also für alle Werte von λ.

Das erste Multiplikationstheorem ist jetzt bewiesen. Den Beweis des zweiten liefert man ganz analog mit Hilfe von (16).

Das dritte Multiplikationstheorem kann aus (17) abgeleitet werden. Genau so wie (26) findet man nämlich, falls $|\eta| < |z|$ ist,

$$(z+\eta)^{-1+a_1}\, G_{p,q}^{m,n}\left(\frac{1}{z+\eta} \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \sum_{r=0}^{\infty} \frac{\eta^r}{r!} \frac{d^r}{(dz)^r} \left\{ z^{-1+a_1}\, G_{p,q}^{m,n}\left(\frac{1}{z} \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)\right\}.$$

Aus (17) ergibt sich aber, falls $n \geqq 1$ ist,

$$\frac{d^r}{(dz)^r}\left\{ z^{-1+a_1}\, G_{p,q}^{m,n}\left(\frac{1}{z} \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right)\right\} = (-1)^r\, z^{-1-r+a_1}\, G_{p,q}^{m,n}\left(\frac{1}{z} \,\middle|\, \begin{matrix} -r+a_1, a_2,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right).$$

Für $|\eta| < |z|$ gilt daher

$$(z+\eta)^{-1+a_1}\, G_{p,q}^{m,n}\left(\frac{1}{z+\eta} \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right) = \sum_{r=0}^{\infty} \frac{(-1)^r \eta^r z^{-1-r+a_1}}{r!}\, G_{p,q}^{m,n}\left(\frac{1}{z} \,\middle|\, \begin{matrix} -r+a_1, a_2,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right). \quad (28)$$

Ist $\left|\frac{1}{\lambda} - 1\right| < 1$, also $\Re(\lambda) > \frac{1}{2}$, so darf man hierin $z = \frac{1}{w}$ und $\eta = \frac{1-\lambda}{\lambda w}$ setzen; Beziehung (28) geht dann in (13) über.

Hiermit ist auch das dritte Multiplikationstheorem bewiesen. Das vierte folgt in ähnlicher Weise aus (18).

§ 5. Anwendungen.

Ich werde (11), (12), (13) und (14) zunächst auf die WHITTAKERschen Funktionen $M_{k,m}(z)$ und $W_{k,m}(z)$ anwenden. Für diese Funktionen gilt[7])

$$e^{\frac{1}{2}z} M_{k,m}(z) = \Gamma(1+2m)\,\Gamma(\tfrac{1}{2}+k-m)\, G_{1,2}^{1,0}\left(z \,\middle|\, \begin{matrix} 1+k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix}\right), \tag{29}$$

$$e^{-\frac{1}{2}z} M_{k,m}(z) = \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+k+m)}\, G_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} 1-k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix}\right), \tag{30}$$

$$e^{\frac{1}{2}z} W_{k,m}(z) = \frac{1}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}\, G_{1,2}^{2,1}\left(z \,\middle|\, \begin{matrix} 1+k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix}\right), \tag{31}$$

$$e^{-\frac{1}{2}z} W_{k,m}(z) = G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1-k \\ \frac{1}{2}+m, \frac{1}{2}-m \end{matrix}\right). \tag{32}$$

Wendet man nun (11) auf (29) an, so bekommt man

$$\lambda^{-m-\frac{1}{2}} e^{\frac{1}{2}\lambda z} M_{k,m}(\lambda z) = \Gamma(1+2m)\,\Gamma(\tfrac{1}{2}+k-m) \sum_{r=0}^{\infty} \frac{(1-\lambda)^r}{r!}\, G_{1,2}^{1,0}\left(z \,\middle|\, \begin{matrix} 1+k \\ \frac{1}{2}+m+r, \frac{1}{2}-m \end{matrix}\right).$$

Aus (24) und (29) folgt aber

$$G_{1,2}^{1,0}\left(z \,\middle|\, \begin{matrix} 1+k \\ \frac{1}{2}+m+r, \frac{1}{2}-m \end{matrix}\right) = z^{\frac{1}{2}r}\, G_{1,2}^{1,0}\left(z \,\middle|\, \begin{matrix} 1+k-\frac{1}{2}r \\ \frac{1}{2}+m+\frac{1}{2}r, \frac{1}{2}-m-\frac{1}{2}r \end{matrix}\right)$$

$$= \frac{1}{\Gamma(1+2m+r)\,\Gamma(\frac{1}{2}+k-m-r)}\, z^{\frac{1}{2}r} e^{\frac{1}{2}z} M_{k-\frac{1}{2}r,\, m+\frac{1}{2}r}(z).$$

Für die Funktion $e^{\frac{1}{2}z} M_{k,m}(z)$ besteht also die Multiplikationsformel

$$\lambda^{-m-\frac{1}{2}} e^{\frac{1}{2}\lambda z} M_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(1+2m)\,\Gamma(\frac{1}{2}+k-m)\,(1-\lambda)^r}{r!\,\Gamma(1+2m+r)\,\Gamma(\frac{1}{2}+k-m-r)}\, z^{\frac{1}{2}r} e^{\frac{1}{2}z} M_{k-\frac{1}{2}r,\, m+\frac{1}{2}r}(z).$$

Durch analoge Anwendung von (11) auf (30) findet man (6).

Die entsprechenden Multiplikationsformeln für die Funktion $W_{k,m}(z)$ sind wegen (31) und (32)

$$\lambda^{-m-\frac{1}{2}} e^{\frac{1}{2}\lambda z} W_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}-k+m+r)\,(1-\lambda)^r}{r!\,\Gamma(\frac{1}{2}-k+m)}\, z^{\frac{1}{2}r} e^{\frac{1}{2}z} W_{k-\frac{1}{2}r,\, m+\frac{1}{2}r}(z)$$

und

$$\lambda^{-m-\frac{1}{2}} e^{-\frac{1}{2}\lambda z} W_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{(1-\lambda)^r}{r!}\, z^{\frac{1}{2}r} e^{-\frac{1}{2}z} W_{k+\frac{1}{2}r,\, m+\frac{1}{2}r}(z);$$

in diesen beiden Relationen wird $|\lambda-1|<1$ vorausgesetzt.

Das zweite Multiplikationstheorem ergibt nach Anwendung auf (29) und (30) die für $|\lambda-1|<1$ gültigen Beziehungen

---

7) Man vergl. [8], Formeln (58) (59), (60) und (61).

$$\lambda^{m-\frac{1}{2}}\, e^{\frac{1}{2}\lambda z}\, M_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(1+2m)(\lambda-1)^r}{r!\,\Gamma(1+2m-r)}\, z^{\frac{1}{2}r}\, e^{\frac{1}{2}z}\, M_{k-\frac{1}{2}r,\,m-\frac{1}{2}r}(z)$$

und

$$\lambda^{m-\frac{1}{2}}\, e^{-\frac{1}{2}\lambda z}\, M_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(1+2m)(\lambda-1)^r}{r!\,\Gamma(1+2m-r)}\, z^{\frac{1}{2}r}\, e^{-\frac{1}{2}z}\, M_{k+\frac{1}{2}r,\,m-\frac{1}{2}r}(z).$$

Die von ERDÉLYI gefundenen Entwicklungen (7) und (8) kommen zum Vorschein, wenn man das dritte Multiplikationstheorem auf (30) bezw. das vierte auf (32) anwendet.

Durch Anwendung des vierten Multiplikationstheorems auf (29) bezw. des dritten auf (31) erhält man

$$\lambda^{-k}\, e^{\frac{1}{2}\lambda z}\, M_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k-m)}{r!\,\Gamma(\frac{1}{2}+k-m-r)} \left(\frac{1}{\lambda}-1\right)^r e^{\frac{1}{2}z}\, M_{k-r,m}(z)$$

bezw.

$$\lambda^{-k}\, e^{\frac{1}{2}\lambda z}\, W_{k,m}(\lambda z) = \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}-k+m+r)\,\Gamma(\frac{1}{2}-k-m+r)}{r!\,\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)} \left(1-\frac{1}{\lambda}\right)^r e^{\frac{1}{2}z}\, W_{k-r,m}(z); \quad (33)$$

Diese zwei Relationen gelten für $\Re(\lambda) > \frac{1}{2}$; (33) ist neuerdings auch von Herrn S. C. DHAR [8]) gefunden worden.

Für die BESSELschen Funktionen $J_\nu(z)$, $Y_\nu(z)$ und $K_\nu(z)$ gilt bekanntlich [9])

$$J_\nu(z) = G^{1,0}_{0,2}\left(\tfrac{1}{4}\, z^2 \,\middle|\, \tfrac{1}{2}\,\nu, -\tfrac{1}{2}\,\nu\right), \quad (34)$$

$$Y_\nu(z) = (-1)^h\, G^{2,0}_{1,3}\left(\tfrac{1}{4}\, z^2 \,\middle|\, \begin{matrix} -\frac{1}{2}\nu-h-\frac{1}{2} \\ \frac{1}{2}\nu, -\frac{1}{2}\nu, -\frac{1}{2}\nu-h-\frac{1}{2} \end{matrix}\right) \quad (h=0, \pm 1, \pm 2, \ldots) \quad (35)$$

und

$$K_\nu(z) = \tfrac{1}{2}\, G^{2,0}_{0,2}\left(\tfrac{1}{4}\, z^2 \,\middle|\, \tfrac{1}{2}\,\nu, -\tfrac{1}{2}\,\nu\right). \quad (36)$$

Die STRUVEsche Funktion

$$\mathbf{H}_\nu(z) = \frac{2^{-\nu-1}\, z^{\nu+1}}{\Gamma(\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})}\, {}_1F_2\left(1; \tfrac{3}{2}, \nu+\tfrac{3}{2}; -\tfrac{1}{4}\, z^2\right)$$

kann folgenderweise in die G-Funktion ausgedrückt werden:

$$\mathbf{H}_\nu(z) = G^{1,1}_{1,3}\left(\tfrac{1}{4}\, z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\nu \\ \frac{1}{2}+\frac{1}{2}\nu, \frac{1}{2}\nu, -\frac{1}{2}\nu \end{matrix}\right). \quad (37)$$

Ich wende nun (12) mit $\frac{1}{4}\, z^2$ statt w und λ^2 statt λ auf (37) an; das Resultat wird, falls $|\lambda^2 - 1| < 1$ ist,

$$\lambda^\nu\, \mathbf{H}_\nu(\lambda z) = \sum_{r=0}^{\infty} \frac{(\lambda^2-1)^r}{r!}\, G^{1,1}_{1,3}\left(\tfrac{1}{4}\, z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\nu \\ \frac{1}{2}+\frac{1}{2}\nu, \frac{1}{2}\nu, -\frac{1}{2}\nu+r \end{matrix}\right).$$

[8]) DHAR, [2], Formel (48).

[9]) Man vergl. [6], 16; [7], 206.

Nun folgt aus (24) und (37), dass

$$G_{1,3}^{1,1}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\nu \\ \frac{1}{2}+\frac{1}{2}\nu,\ \frac{1}{2}\nu,\ -\frac{1}{2}\nu+r \end{matrix}\right)$$

$$=(\tfrac{1}{2}z)^r\, G_{1,3}^{1,1}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}r \\ \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}r,\ \frac{1}{2}\nu-\frac{1}{2}r,\ -\frac{1}{2}\nu+\frac{1}{2}r \end{matrix}\right)=(\tfrac{1}{2}z)^r\, \mathbf{H}_{\nu-r}(z)$$

ist. Für die Funktion $\mathbf{H}_\nu(z)$ gilt also, falls $|\lambda^2-1|<1$ ist,

$$\lambda^\nu\, \mathbf{H}_\nu(\lambda z)=\sum_{r=0}^{\infty}\frac{(\lambda^2-1)^r\,(\frac{1}{2}z)^r}{r!}\,\mathbf{H}_{\nu-r}(z).$$

Formel (2) kann genau so mit Hilfe von (12) aus (34) abgeleitet werden. Die Beziehungen (1), (3), (4) und (5) kommen zum Vorschein, wenn man das erste Multiplikationstheorem auf (34), (35) und (36) anwendet.

Zum Schluss werde ich (11) und (12) noch anwenden auf die LOMMEL-schen Funktionen[10]) $s_{\mu,\nu}(z)$ und $S_{\mu,\nu}(z)$. Auch diese Funktionen sind Spezialfälle der G-Funktion; man hat nämlich[11])

$$s_{\mu,\nu}(z)=2^{\mu-1}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)\,G_{1,3}^{1,1}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\mu \\ \frac{1}{2}+\frac{1}{2}\mu,\ \frac{1}{2}\nu,\ -\frac{1}{2}\nu \end{matrix}\right) . \quad (38)$$

und

$$S_{\mu,\nu}(z)=\frac{2^{\mu-1}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)}\,G_{1,3}^{3,1}\left(\tfrac{1}{4}z^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}\mu \\ -\frac{1}{2}\nu,\ \frac{1}{2}\nu,\ \frac{1}{2}+\frac{1}{2}\mu \end{matrix}\right) . \quad (39)$$

Wendet man nun (12) mit $\frac{1}{4}z^2$ statt w und λ^2 statt λ auf (38) an, so bekommt man mit Rücksicht auf (24)

$$\lambda^\nu\, s_{\mu,\nu}(\lambda z)=\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)\,(\lambda^2-1)^r}{r!\,\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu-r)}\,z^r\, s_{\mu-r,\nu-r}(z) \quad . \quad . \quad (40)$$

Genau so findet man durch Anwendung von (11) auf (39)

$$\lambda^\nu\, S_{\mu,\nu}(\lambda z)=\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu+r)\,(1-\lambda^2)^r}{r!\,\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)}\,z^r\, S_{\mu-r,\nu-r}(z). \quad (41)$$

Die Beziehungen (40) und (41) gelten für $|\lambda^2-1|<1$.

LITERATURVERZEICHNIS.

E. W. BARNES.

1. The asymptotic expansion of integral functions defined by generalized hypergeometric series. Proc. London Math. Soc., (2) **5**, 59—116 (1907).

S. C. DHAR.

2. On operational representations of confluent hypergeometric functions and their integrals. Philosophical Magazine, (7) **21**, 1082—1096 (1936).

A. ERDÉLYI.

3. Funktionalrelationen mit konfluenten hypergeometrischen Funktionen. Erste

[10]) Für die Definition der Funktionen $s_{\mu,\nu}(z)$ und $S_{\mu,\nu}(z)$ siehe man WATSON, [**10**], 345—349.

[11]) Man vergl. [**7**], 205—206.

Mitteilung: Additions- und Multiplikationstheoreme. Math. Zeitschrift, **42**, 125—143 (1937).

4. Ueber eine Methode zur Gewinnung von Funktionalbeziehungen zwischen konfluenten hypergeometrischen Funktionen. Monatshefte für Math. und Phys., **45**, 31—52 (1937).

S. GOLDSTEIN.

5. Operational representations of WHITTAKER's confluent hypergeometric function and WEBER's parabolic cylinder function. Proc. London Math. Soc., (2) **34**, 103—125 (1932).

C. S. MEIJER.

6. Ueber WHITTAKERsche bezw. BESSELsche Funktionen und deren Produkte. Nieuw Archief voor Wiskunde, (2) **18**, (4tes Heft) 10—39 (1936).

7. Ueber BESSELsche, STRUVEsche und LOMMELsche Funktionen. Proc. Kon. Akad. v. Wetensch., Amsterdam, **43**, 198—210 und 366—378 (1940).

8. Neue Integraldarstellungen für WHITTAKERsche Funktionen. Ibidem, **44**, 81—92, 186—194, 298—307, 442—451 und 590—598 (1941).

9. Integraldarstellungen für WHITTAKERsche Funktionen und ihre Produkte. Ibidem, **44**, 435—441 und 599—605 (1941).

G. N. WATSON.

10. A treatise on the theory of BESSEL functions (1922).

Mathematics. — *Ein Satz über die sukzessiven Minima bei definiten quadratischen Formen.* Von ROBERT REMAK. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of October 25, 1941.)

Es sei

$$y_i = \sum_{k=1}^{n} c_{ik}\, x_k \qquad (i = 1.\, 2, \ldots, n) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

ein System von n Linearformen in n Veränderlichen. Es sei die Determinante

$$d = |\, c_{ik}\,| \neq 0.$$

Betrachtet man die y_i als kartesische Koordinaten und lässt man für die x_k nur ganze rationale Werte zu, so stellen die y_i ein, im allgemeinen schiefwinkliges, Punktgitter im n-dimensionalen Raume dar. *abs d* ist das Volumen des Elementarparallelepipeds. Es bezeichne K einen konvexen Körper vom Volumen V, dessen Mittelpunkt im 0-Punkte des Koordinatensystems gelegen sei. Der bekannte Satz von MINKOWSKI sagt aus, dass K ausser dem 0-Punkte weitere Gitterpunkte enthalten muss, wenn

$$V \geqq 2^n\, d,$$

oder anders formuliert: Es sei K ein konvexer Körper, dessen Mittelpunkt im 0-Punkt liege. Es sei das Volumen V von K beliebig, λK, wo $\lambda > 0$, sei der im Verhältnis $1 : \lambda$ vom 0-Punkte aus ähnlich vergrösserte Körper, dessen Volumen $\lambda^n V$ ist. Für hinreichend kleine λ enthält λK ausser dem 0-Punkte keinen anderen Gitterpunkt. Vergrössert man λ bis zu dem kleinsten Werte λ_1, für den $\lambda_1 K$ einen weiteren Gitterpunkt enthält, so ist

$$\lambda_1^n\, V \leqq 2^n \,.\, d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Diesen Satz hat MINKOWSKI folgendermassen verallgemeinert. Es sei P_1 der erste Gitterpunkt, der durch Vergrösserung auf $\lambda_1 K$ erreicht wird. (Natürlich wird gleichzeitig der zum 0-Punkt O diametrale Gitterpunkt $-P_1$ erreicht.) Man nehme die Gerade $O P_1$ aus und vergrössere der Körper so lange weiter, bis im Körper $\lambda_2 K$ ein weiterer Gitterpunkt P_2 erreicht wird. Es ist $\lambda_2 \geqq \lambda_1$. ($\lambda_2 = \lambda_1$ ist nicht ausgeschlossen). Dann nehme man die Ebene $O P_1 P_2$ aus und bestimme λ_3, so dass mit $\lambda_3 K$ zum ersten Male einer der übrigen Gitterpunkte erreicht wird, bis man n unabhängige Gitterpunkte $P_1 P_2, \ldots . P_n$ und n Zahlenwerte $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ gefunden hat. Dann beweist MINKOWSKI [1])

$$\lambda_1\, \lambda_2 \ldots \lambda_n \,.\, V \leqq 2^n\, d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Da $\lambda_1^n \,.\, V \leqq \lambda_1 \lambda_2 \ldots \lambda_n \,.\, V$, so ist (2) eine Folge von (3). Dagegen sagt (3) mehr aus als (2).

Eine andere Verschärfung des MINKOWSKIschen Satzes ist folgende. Der konvexe Körper K sei speziell die Kugel mit dem Radius 1, d.h. das Gebiet

$$y_1^2 + y_2^2 + \ldots + y_n^2 \leqq 1.$$

---

[1]) MINKOWSKI, Geometrie der Zahlen, Leipzig 1910, S. 187—219.

Diesen Satz hat Herr DAVENPORT einfacher bewiesen: Quarterly Journal of Mathematics, Oxford Series 10, (1938) p. 119—121.

Dessen Volumen ist bekanntlich

$$\frac{\pi^{\frac{n}{2}}}{\Gamma\left(1+\frac{n}{2}\right)} = \varkappa_n.$$

Man bilde aus (1) die quadratische Form

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} c_{ik}\, x_k\right)^2 = \sum_{k,l}^{n} a_{kl}\, x_k\, x_l, \quad . \quad (4)$$

worin

$$a_{kl} = \sum_{i=1}^{n} c_{ik}\, c_{il}.$$

Oder man schreibe Matrizen. Es sei

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

eine Spalte, ebenso Y eine Spalte von y_i.

$$A = (a_{ik}) \quad ; \quad C = (c_{ik}).$$

C' sei die transponierte Matrix, die aus C durch Vertauschung von Zeilen und Spalten entsteht.

$$C' = (c_{ki}).$$

Dann lautet (1)

$$Y = C\,X.$$

Ferner lautet (4)

$$f(X) = Y'\,Y = X'\,C'\,C\,X = X'\,A\,X.$$

Ferner ist nach dem Multiplikationstheorem der Determinanten

$$a_{ik} = |A| = |C'\,C| = |C'|\,|C| = |C|^2 = d^2 > 0$$

die Determinante der quadratischen Form.

Ist m_1 der kleinste Wert, den die quadratische Form f für ganzzahlige x_ν annimmt, die nicht sämtlich 0 sind, so ist

$$\lambda_1 = \sqrt{m_1}.$$

Die Kugel mit dem Radius $\sqrt{m_1}$ ist die kleinste Kugel, die einen Gitterpunkt ausser dem 0-Punkt enthält. Der MINKOWSKIsche Satz sagt aus nach (2), worin $V = \varkappa_n$ zu setzen ist,

$$\lambda_1^n \,.\, V = m_1^{\frac{n}{2}} \,.\, V \leqq 2^n \,.\, d,$$

oder quadriert

$$\frac{\varkappa_n^2}{2^{2n}} = \frac{\pi^n}{2^{2n}\left(\Gamma\left(1+\frac{n}{2}\right)\right)^2} \leqq \frac{|A|}{m_1^n}.$$

Diese Ungleichung ist durch Herrn BLICHFELDT [2]) wesentlich verbessert

$$\gamma_n = \frac{\pi^n}{2^n \left(\Gamma\left(2 + \frac{n}{2}\right)\right)^2} \leqq \frac{|A}{m_1^n}.$$

Ich will im folgenden zeigen, dass man die beiden Verschärfungen des MINKOWSKIschen Satzes mit einander kombinieren kann. Zu dem Zwecke sollen die sukzessiven Minima der quadratischen Form gebildet werden. Es sei

$$u_{11}, u_{21}, \ldots, u_{n1} \quad \ldots\ldots\ldots\ldots \quad (5)$$

ein System ganzer Zahlen, für das die quadratische Form f ihren kleinst möglichen von 0 verschiedenen Wert m_1 annimmt. Man nehme die Vielfachen von $u_{11}, u_{21}, \ldots u_{n1}$ aus und bestimme das Minimum m_2 der quadratischen Form unter den übrigen ganzzahligen Werten der x_ν, das für

$$u_{12}, u_{22}, \ldots, u_{n2} \quad \ldots\ldots\ldots\ldots \quad (6)$$

angenommen werde. Dann nehme man die linearen Verbindungen von (5) und (6) aus, bestimme das dritte Minimum und so fort, bis man n Minima bestimmt hat. Man bilde die Matrix

$$\begin{pmatrix} u_{11}, u_{12}, \ldots, u_{1n} \\ u_{21}, u_{22}, \ldots, u_{2n} \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ u_{n1}, u_{n2}, \ldots, u_{nn} \end{pmatrix} = U,$$

in der die Systeme (5) und (6) die erste und zweite Spalte bilden. Wegen der linearen Unabhängigkeit der Spalten ist die Determinante

$$|U| \neq 0.$$

Für $n \geqq 4$ kann $abs\,|U| > 1$ sein. Das Verfahren liefert also nicht immer ein Elementarparallelepiped. Ich habe es daher [3]) als „Pseudoreduktion nach sukzessiven Minima" bezeichnet. Es möge $|U|$ die Verschränkungsdeterminante heissen. Für diese werde ich eine einfache Abschätzung angeben.

Es sei

$$m_\nu = f(u_{1\nu}, u_{2\nu}, \ldots, u_{n\nu}) \quad \ldots\ldots\ldots\ldots \quad (7)$$

das ν-te Minimum. Es ist

$$m_\nu \leqq m_{\nu+1} \quad \ldots\ldots\ldots\ldots \quad (8)$$

da m_ν das Minimum der umfassenderen Menge ist. Dann will ich folgenden Satz beweisen:

---

[2]) BLICHFELDT, Transactions of the American Mathematical Society 15 (1914), S. 227—235. REMAK, Math. Zeitschr. 26, (1927), S. 694—699. BLICHFELDT, Math. Annalen 101 (1929), S. 605—608. In der letztgenannten Arbeit vergrössert Herr BLICHFELDT γ_n um den Zusatzfaktor $(1+g)^2$ worin

$$g = \frac{1}{2^{\frac{n}{2}}} \left\{ \frac{1}{n+1} - (2-\sqrt{2})^{n+1} \left(\frac{\sqrt{2}}{2} + \frac{1}{n+1} \right) \right\},$$

der für grosse n gegen 1 konvergiert.

[3]) REMAK, Ueber die MINKOWSKIsche Reduktion der definiten quadratischen Formen. Compositio Mathematica, vol. 5 (1938), S. 368—391. Siehe S. 368.

Es sei für eine positive Zahl γ_n' allgemein für alle definiten quadratischen Formen von n Veränderlichen bewiesen, dass

$$\gamma_n' \leqq \frac{|A|}{m_1^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Dann gilt auch für alle definiten quadratischen Formen von n Veränderlichen

$$\gamma_n' \leqq \frac{|A|}{m_1, m_2 \ldots m_n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

und

$$abs\,|U| \leqq \gamma_n'^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)\ [4])$$

Die Buchstaben haben hierin die obige Bedeutung.

Für γ_n' kann man die BLICHFELDTschen γ_n und für die kleineren Werte von n die genauen Werte, soweit sie bekannt sind, benutzen.[5])

Beweis: Man setze:

$$U' A U = \bar{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$U^{-1} X = X. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Für $\bar{X}$ lasse man alle einspaltigen Wertsysteme rationaler Zahlen zu, die sich aus (13) ergeben, wenn X jedes beliebige einspaltige ganzzahlige Wertsystem sein darf. Wenn X ganzzahlig, ist auch UX ganzzahlig. Also kommen unter den Wertsystemen $\bar{X}$ in (13) auch die Systeme

$$U^{-1} U X = X$$

d.h. alle ganzzahligen Wertsysteme vor. Es gilt nach (12) für die Determinante

$$|U|^2 . |A| = |\bar{A}| . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Man bilde

$$f(X) = \bar{X}' A \bar{X} = (X' U'^{\,1})(U' A U)(U^{-1} X) = X' A X = f(X).$$

Es ist der v-te Koeffizient von $\bar{f}$ in der Hauptdiagonale mit Rücksicht auf (7)

$$\bar{a}_{vv} = \sum_{i,k=1}^{n} u_{iv}\, a_{ik}\, u_{kv} = f(u_{1v}, u_{2v}, \ldots, u_{nv}) = m_v . \quad . \quad . \quad (15)$$

Es ist

$$\bar{a}_{vv} = f(\bar{x}_1 = 0; \bar{x}_2 = 0; \ldots; \bar{x}_v = 1; \bar{x}_{v+1} = 0; \ldots \bar{x}_n = 0).$$

Die $v-1$-dimensionale Ebene der ersten $v-1$ Minima wird bestimmt durch

$$\bar{x}_v = 0;\ \bar{x}_{v+1} = 0; \ldots \bar{x}_n = 0$$

---

[4]) Nach MINKOWSKI, l.c. Anm. [1]) S. 189 ist im Falle der sukzessiven Minima für einen beliebigen konvexen Körper mit Mittelpunkt die Verschränkungsdeterminante, die dort N heisst,

$$N < n!$$

[5]) $\gamma_2 = \frac{3}{4}$; $\gamma_3 = \frac{1}{2}$; $\gamma_4 = \frac{1}{4}$; $\gamma_5 = \frac{1}{8}$; $\gamma_6 = \frac{3}{64}$; $\gamma_7 = \frac{1}{64}$; $\gamma_8 = \frac{1}{256}$.

Litteratur siehe Diophantische Approximationen. J. F. KOKSMA, Berlin, 1936, S. 24.

Es ist also

$$\bar{a}_{\nu\nu} = Min\, \bar{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

unter der Bedingung, dass

$$\bar{x}_\nu, \bar{x}_{\nu+1}, \ldots \bar{x}_n$$

nicht sämtlich 0 sein dürfen. Die für $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n$ zulässigen Wertsysteme werden durch (13) bei ganzzahligem X bestimmt. Es ist wegen (15) und (8)

$$\bar{a}_{11} \leqq \bar{a}_{22} \leqq \ldots \leqq \bar{a}_{nn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16a)$$

Man führe in $\bar{A}$ die Staffelzerlegung aus. Es sei also

$$\bar{A} = B' \bar{B}.$$

Hierin sei

$$B = (\bar{b}_{h\nu}),$$

worin

$$\bar{b}_{h\nu} = 0 \qquad \text{für} \quad h > \nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Es ist

$$|\bar{A}| = |\bar{B}|^2 = \bar{b}_{11}^2 . \bar{b}_{22}^2, \ldots, \bar{b}_{nn}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

ferner

$$\bar{a}_{\nu\nu} = \sum_{h=1}^{n} \bar{b}_{h\nu}^2 \geqq \bar{b}_{\nu\nu}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Aus (18) und (19) folgt

$$|\bar{A}| \leqq \bar{a}_{11} . \bar{a}_{22} \ldots \bar{a}_{nn}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Im folgenden soll eine Hilfsform gebildet werden. Man setze

$$\left.\begin{aligned} \bar{b}_{h\nu}^* &= \frac{b_{h\nu}}{\sqrt{\bar{a}_{hh}}}, \\ \bar{B}^* &= (\bar{b}_{h\nu}^*), \\ \bar{A}^* &= \bar{B}^{*\prime}\, \bar{B}^*. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Es ist wegen (17) auch $\bar{B}^*$ eine Staffelmatrix. Es gilt analog (18)

$$\left.\begin{aligned} |\bar{A}^*| &= |\bar{B}^{*\,2}| = \bar{b}_{11}^{*2} . \bar{b}_{22}^{*2}, \ldots, \bar{b}_{nn}^{*2} \\ &= \frac{b_{11}^2}{\bar{a}_{11}} . \frac{\bar{b}_{22}^2}{\bar{a}_{22}}, \ldots, \frac{\bar{b}_{nn}^2}{\bar{a}_{nn}} = \frac{|\bar{A}|}{\bar{a}_{11}, \bar{a}_{22} \ldots \bar{a}_{nn}}. \end{aligned}\right\} \quad . \quad . \quad (22)$$

Also ist wegen (20)

$$|\bar{A}^*| \leqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Für $\bar{X}$ werden dieselben Werte wie vorher zugelassen. Es soll gezeigt werden, dass, wenn nicht sämtliche $\bar{x}_\nu = 0$ sind,

$$f^*(\bar{X}) = X' \bar{A}^* X \leqq 1. \; [6]). \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

---

[6]) Aehnliche Schlüsse habe ich bereits l.c. Anm. [3]) S. 381 angewandt.

Es sei s der grösste Index, für den

$$\bar{x}_s \neq 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Dann ist analog (4) unter Benutzung von (21), (25), (17), (16a), (16)

$$\left.\begin{aligned} \bar{f}^*(\bar{X}) &= \bar{X}'\,\bar{A}^*\,\bar{X} = (\bar{X}'\,\bar{B}^{*\prime})\,(\bar{B}^*\,\bar{X}) \\ &= \sum_{h=1}^{n}\left(\sum_{i=1}^{n}\bar{b}^*_{hi}\,\bar{x}_i\right)^2 = \sum_{h=1}^{n}\frac{\left(\sum_{i=1}^{n}\bar{b}_{hi}\,\bar{x}_i\right)^2}{\bar{a}_{hh}} \\ &= \sum_{h=1}^{n}\frac{\left(\sum_{i=1}^{s}\bar{b}_{hi}\,\bar{x}_i\right)^2}{\bar{a}_{hh}} = \sum_{h=1}^{s}\frac{\left(\sum_{i=1}^{s}\bar{b}_{hi}\,\bar{x}_i\right)^2}{\bar{a}_{hh}} \geqq \frac{\sum_{h=1}^{s}\left(\sum_{i=1}^{s}\bar{b}_{hi}\,\bar{x}_i\right)}{\bar{a}_{ss}} \\ &= \frac{\bar{f}(\bar{X})}{\bar{a}_{ss}} \geqq 1. \end{aligned}\right\} \quad (26)$$

Da dies für alle Indizes s gilt, ist (24) bewiesen. Zur besseren Uebersicht über (26) mögen folgende Bemerkungen dienen. Betrachtet man die h-te Linearform, die im vierten Ausdruck als Basis eines Quadrats auftritt, $\sum_{i=1}^{n}\bar{b}^*_{hi}\,x_i$, so enthält sie wegen (17) nur die x_i mit $i \geqq h$. Wegen (25) braucht i nur bis s zu laufen. Dadurch fallen aber die letzten Linearformen mit $h > s$ weg, da ihre sämtlichen Glieder 0 sind. Diese Betrachtungen gelten wegen (21) sowohl in $\bar{f}(X)$ als in $\bar{f}^*(\bar{X})$. Also darf die nur bis s summierte Doppelsumme im drittletzten Ausdruck von (26) gleich $\bar{f}(\bar{X})$ gesetzt werden.

Man setze

$$f^*(X) = \bar{f}^*(\bar{X}) = \bar{X}'\,\bar{A}^*\,\bar{X} = (X'\,U'^{-1})\,\bar{A}^*\,(U^{-1}\,X) = X'\,A^*\,X. \quad (27)$$

Also ist

$$A^* = U'^{-1}\,\bar{A}^*\,U^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

m^* sei das Minimum von $f^*(X)$ für ganzzahlige X, wenn nicht sämtliche $x_\nu = 0$. Wegen (24) und (27) ist

$$m^* \geqq 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Es ist wegen (28)

$$|A^*| = |\bar{A}^*|\,.\,|U|^{-2},$$

also nach (9), (29), (22), (14), (15)

$$\gamma'_n \leqq \frac{|A^*|}{m^{*}{}_n} \leqq |A^*| = |\bar{A}^*|\,.\,|U|^{-2} = \frac{|\bar{A}|\,.\,|U|^{-2}}{\bar{a}_{11}\,.\,\bar{a}_{22}\,.\,\bar{a}_{nn}} = \frac{|A|}{m_1\,.\,m_2\,.\,.\,.\,m_n}. \quad (30)$$

Damit ist (10) bewiesen.

Ferner folgt aus (30) und (23)

$$\gamma'_n \geqq |A^*|\,.\,|U|^{-2} \geqq |U|^{-2},$$

also

$$abs\,|U| \leqq \gamma'^{-\frac{1}{2}}_n.$$

Damit ist auch (11) bewiesen.

Mathematics. — *Stark konvergente Entwicklungen für die vollständigen elliptischen Integrale erster und zweiter Art.* II. Von S. C. VAN VEEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 25, 1941.)

§ 2. *k liegt in der Nähe von 1.*

Hilfssatz 1: Für ganzes $k \geqq 0$, $|1-z| < 1$, $|arg\,(1-z)| < \pi$ ist

$$\left.\begin{aligned}
&\Gamma(a+k)\,.\,\Gamma(b+k)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,F(a,b;a+b+k;z) = \\
&= \Gamma(a+b+k)\sum_{n=0}^{k-1}(-1)^n\frac{(k-n-1)!\,\Gamma(a+n)\,\Gamma(b+n)}{n!}(1-z)^n + \\
&+(-1)^k\,\Gamma(a+b+k)\sum_{n=k}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)}{n!\,(n-k)!} \\
&\left\{\frac{\Gamma'(1-k+n)}{\Gamma(1-k+n)}+\frac{\Gamma'(1+n)}{\Gamma(1+n)}-\frac{\Gamma'(a+n)}{\Gamma(a+n)}-\frac{\Gamma'(b+n)}{\Gamma(b+n)}-\log(1-z)\right\}(1-z)^n
\end{aligned}\right\} \quad (19)$$

(Die erste Summe im rechten Gliede $= 0$ für $k = 0$.)

Beweis: Im Innern der von $z = 1$ nach $+\infty$ aufgeschlitzten Ebene gilt, *wenn c—a—b keine ganze Zahl oder 0 ist* [1])

$$\begin{aligned}
&\Gamma(c-a)\,.\,\Gamma(c-b)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,F(a,b;c;z) = \\
&= \Gamma(c)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,\Gamma(c-a-b)\,.\,F(a,b;a+b-c+1;1-z) + \\
&+\Gamma(c)\,.\,\Gamma(c-a)\,.\,\Gamma(c-b)\,.\,\Gamma(a+b-c)\,.\,(1-z)^{c-a-b}\,.\,F(c-a,c-b;c-a-b+1;1-z)
\end{aligned}$$

Hierin setzen wir

$$c = a+b+k+\delta, \quad \text{mit} \quad |\delta| < 1.$$

Dann ist

$$\left.\begin{aligned}
&\Gamma(b+k+\delta)\,.\,\Gamma(a+k+\delta)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,F(a,b;a+b+k+\delta;z) = \\
&= \Gamma(a+b+k+\delta)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,\Gamma(k+\delta)\,.\,F(a,b;1-k-\delta;1-z) + \\
&+\Gamma(a+b+k+\delta)\,.\,\Gamma(b+k+\delta)\,.\,\Gamma(a+k+\delta)\,.\,\Gamma(-k-\delta)\,.\,(1-z)^{k+\delta}\,. \\
&.\,F(b+k+\delta,a+k+\delta;1+k+\delta;1-z).
\end{aligned}\right\} \quad (20)$$

Wegen

$$|1-z| < 1$$

geht (20) über in

$$\begin{aligned}
&\Gamma(b+k+\delta)\,.\,\Gamma(a+k+\delta)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,F(a,b;a+b+k+\delta;z) = \\
&= \Gamma(a+b+k+\delta)\,.\,\Gamma(k+\delta)\,.\,\Gamma(1-k-\delta)\,.\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{\Gamma(1-k-\delta+n)\,.\,n!}(1-z)^n \\
&+\Gamma(a+b+k+\delta)\,.\,\Gamma(k+\delta+1)\,.\,\Gamma(-k-\delta)\,.\sum_{n=0}^{\infty}\frac{\Gamma(a+k+\delta+n)\,.\,\Gamma(b+k+\delta+n)}{\Gamma(1+k+\delta+n)\,.\,n!}(1-z)^{n+k+\delta} = \\
&=(-1)^k\,\Gamma(a+b+k+\delta)\,.\,\frac{\pi}{\sin\pi\delta}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(1-k-\delta+n)\,.\,n!}(1-z)^n + \\
&+(-1)^{k+1}\,\Gamma(a+b+k+\delta)\,.\,\frac{\pi}{\sin\pi\delta}\,.\sum_{n=0}^{\infty}\frac{\Gamma(a+k+\delta+n)\,.\,\Gamma(b+k+\delta+n)}{\Gamma(1+k+\delta+n)\,.\,n!}(1-z)^{n+k+\delta},
\end{aligned}$$

[1]) GAUSS, Werke III, S. 213, Formel 87; vgl. auch WHITTAKER–WATSON, Modern Analysis, 2nd Ed., § 14, 53.

also
$$\Gamma(b+k+\delta)\,.\,\Gamma(a+k+\delta)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,F(a,b;a+b+k+\delta;z)=$$
$$=(-1)^k\,\Gamma(a+b+k+\delta)\,\frac{\pi}{\sin\pi\delta}\cdot\sum_{n=0}^{k-1}\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{\Gamma(1-k-\delta+n)\,.\,n!}(1-z)^n+$$
$$+(-1)^k\,\Gamma(a+b+k+\delta)\,\frac{\pi}{\sin\pi\delta}\sum_{n=k}^{\infty}\left\{\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{\Gamma(1-k-\delta+n)\,.\,n!}(1-z)^n\right.$$
$$\left.-\frac{\Gamma(a+\delta+n)\,.\,\Gamma(b+\delta+n)}{\Gamma(1+\delta+n)\,.\,(n-k)!}(1-z)^{n+\delta}\right\} \tag{21}$$

(Die erste Summe im rechten Gliede $=0$ für $k=0$.) Wir betrachten

$$f(\delta)=\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{\Gamma(1-k-\delta+n)\,.\,n!}-\frac{\Gamma(a+\delta+n)\,.\,\Gamma(b+\delta+n)}{\Gamma(1+\delta+n)\,.\,(n-k)!}(1-z)^{\delta}$$

und setzen ohne Beschränkung der Allgemeinheit δ als reell voraus [1]), also

$$f(\delta)=f(0)+\delta\,f'(\theta\,\delta)\qquad(0\leqq\theta\leqq 1)$$
$$=\delta\left\{\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{\Gamma(1-k-\theta\delta+n)\,.\,n!}\cdot\frac{\Gamma'(1-k-\theta\delta+n)}{\Gamma(1-k-\theta\delta+n)}+\right.$$
$$-\frac{\Gamma(a+\theta\,\delta+n)\,.\,\Gamma(b+\theta\,\delta+n)}{\Gamma(1+\theta\delta+n)\,.\,(n-k)!}(1-z)^{\delta}\left(\frac{\Gamma'(a+\theta\delta+n)}{\Gamma(a+\theta\delta+n)}+\right.$$
$$\left.\left.+\frac{\Gamma'(b+\theta\delta+n)}{\Gamma(b+\theta\delta+n)}-\frac{\Gamma'(1+\theta\delta+n)}{\Gamma(1+\theta\delta+n)}+\log(1-z)\right)\right\}. \tag{22}$$

Aus (21) und (22) ergibt sich

$$\Gamma(b+k+\delta)\,.\,\Gamma(a+k+\delta)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,F(a,b;a+b+k+\delta;z)=$$
$$=(-1)^k\,\Gamma(a+b+k+\delta)\,\frac{\pi}{\sin\pi\delta}\cdot\sum_{n=0}^{k-1}\frac{\sin\pi(k+\delta-n)\,.\,\Gamma(a+n)\,.\,\Gamma(b+n)\,.\,\Gamma(k+\delta-n)}{\pi\,.\,n!}(1-z)^n+$$
$$+(-1)^k\,.\,\Gamma(a+b+k+\delta)\,.\,\frac{\pi\delta}{\sin\pi\delta}\cdot\sum_{n=k}^{\infty}\left\{\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{\Gamma(1-k-\theta\delta+n)\,.\,n!}\cdot\frac{\Gamma'(1-k-\theta\delta+n)}{\Gamma(1-k-\theta\delta+n)}+\right.$$
$$-\frac{\Gamma(a+\theta\delta+n)\,.\,\Gamma(b+\theta\delta+n)}{\Gamma(1+\theta\delta+n)\,.\,(n-k)!}(1-z)^{\delta}\left(\frac{\Gamma'(a+\theta\delta+n)}{\Gamma(a+\theta\delta+n)}+\right.$$
$$\left.\left.+\frac{\Gamma'(b+\theta\delta+n)}{\Gamma(b+\theta\delta+n)}-\frac{\Gamma'(1+\theta\delta+n)}{\Gamma(1+\theta\delta+n)}+\log(1-z)\right)\right\}(1-z)^n$$

und schliesslich, für $\delta\to 0$

$$\Gamma(b+k)\,.\,\Gamma(a+k)\,.\,\Gamma(a)\,.\,\Gamma(b)\,.\,F(a,b;a+b+k;z)=$$
$$=\Gamma(a+b+k)\,.\sum_{n=0}^{k-1}(-1)^n\,.\,\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{n!}\,\Gamma(k-n)(1-z)^n+$$
$$+(-1)^k\,.\,\Gamma(a+b+k)\,.\sum_{n=k}^{\infty}\frac{\Gamma(a+n)\,.\,\Gamma(b+n)}{n!\,(n-k)!}\left\{\frac{\Gamma'(1-k+n)}{\Gamma(1-k+n)}+\right.$$
$$\left.+\frac{\Gamma'(1+n)}{\Gamma(1+n)}-\frac{\Gamma'(a+n)}{\Gamma(a+n)}-\frac{\Gamma'(b+n)}{\Gamma(b+n)}-\log(1-z)\right\}(1-z)^n$$

w. z. b. w.

[1]) (Zusatz bei der Korrektur.) Zunächst sei auch a, b und z reell, mit $a>1-k$, $b>1-k$, $0<z<1$. Analytische Fortsetzung gibt das allgemeine Ergebnis (19) für komplexes a, b und z.

Bemerkung: Der Beweis dieses wichtigen Satzes ist im wesentlichen aus GAUSS (Werke III, S. 214 u. f.) entlehnt.

Im folgenden werden nur die zwei Sonderfälle

$$A) \quad a=\tfrac{1}{2},\ b=\tfrac{1}{2},\ k=0,$$

$$B) \quad a=-\tfrac{1}{2},\ b=\tfrac{1}{2},\ k=1$$

verwendet. Wegen

$$\int_0^1 \frac{x^{m-1}(1-x^p)}{1-x}\,dx=\lim_{\delta\to 0}\int_0^1 \frac{x^{m-1}(1-x^p)}{(1-x)^{1-\delta}}\,dx=\lim_{\delta\to 0}\left\{\frac{\Gamma(m)\,\Gamma(\delta)}{\Gamma(m+\delta)}-\frac{\Gamma(m+p)\,\Gamma(\delta)}{\Gamma(m+p+\delta)}\right\}=$$

$$=\lim_{\delta\to 0}\left\{\frac{\Gamma(m+p+\delta)-\Gamma(m+p)}{\delta}\cdot\frac{\Gamma(1+\delta)}{\Gamma(m+p+\delta)}-\frac{\Gamma(m+\delta)-\Gamma(m)}{\delta}\cdot\frac{\Gamma(1+\delta)}{\Gamma(m+\delta)}\right\}=\frac{\Gamma'(m+p)}{\Gamma(m+p)}-\frac{\Gamma'(m)}{\Gamma(m)}$$

für $R(m)>0$, $R(m+p)>0$, findet man für ganzes $n\geqq 0$

$$\frac{\Gamma'(1+n)}{\Gamma(1+n)}-\frac{\Gamma'(\frac{1}{2}+n)}{\Gamma(\frac{1}{2}+n)}=\int_0^1 \frac{x^{n-\frac{1}{2}}(1-x^{\frac{1}{2}})}{1-x}\,dx=2\log 2+2\sum_{k=1}^{2n}\frac{(-1)^k}{k}=2\log 2-2\sum_{m=1}^{n}\frac{1}{(2m-1)\,2m}.$$

$$A) \quad \underline{a=\tfrac{1}{2},\ b=\tfrac{1}{2},\ k=0}$$

$$\left.\begin{aligned}\pi^2 F(\tfrac{1}{2},\tfrac{1}{2};1;z)&=\sum_{n=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{2}+n)}{n!}\right\}^2(1-z)^n\left\{4\log 2-4\sum_{m=1}^{n}\frac{1}{(2m-1)\,2m}-\log(1-z)\right\}=\\ &=\pi\cdot\log\frac{16}{1-z}\cdot F(\tfrac{1}{2},\tfrac{1}{2};1;1-z)-4\sum_{n=0}^{\infty}\left\{\frac{\Gamma(\frac{1}{2}+n)}{n!}\right\}^2(1-z)^n\sum_{m=1}^{n}\frac{1}{(2m-1)\,2m}.\end{aligned}\right\}\quad(23)$$

(vgl. GAUSS, Werke III, S. 217, [90]).

$$B) \quad \underline{a=-\tfrac{1}{2},\ b=\tfrac{1}{2},\ k=1}$$

$$\left.\begin{aligned}-\pi^2 F(-\tfrac{1}{2},\tfrac{1}{2};1;z)&=-2\pi-\sum_{n=1}^{\infty}\frac{\Gamma(n-\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{n!\,(n-1)!}\left\{4\log 2-4\sum_{m=1}^{n}\frac{1}{(2m-1)\,2m}+\right.\\ &\left.+\frac{1}{n(2n-1)}-\log(1-z)\right\}(1-z)^n=-2\pi\cdot F(-\tfrac{1}{2},-\tfrac{1}{2};1;1-z)\\ &-\frac{\pi}{2}(1-z)\log\frac{16}{1-z}F(\tfrac{1}{2},\tfrac{3}{2};2;1-z)+4\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{3}{2})}{(n+1)!\,n!}(1-z)^{n+1}\sum_{m=1}^{n+1}\frac{1}{(2m-1)\,2m}.\end{aligned}\right\}\quad(24)$$

Schreibt man

$$\sum_{n=q}^{\infty}a_n=S_q$$

so ist

$$\sum_{n=0}^{\infty}a_n\sum_{m=1}^{n}b_m=\sum_{n=0}^{\infty}(S_n-S_{n+1})\sum_{m=1}^{n}b_m=\sum_{n=1}^{\infty}S_n\Big(\sum_{m=1}^{n}b_m-\sum_{m=1}^{n-1}b_m\Big)=\sum_{n=1}^{\infty}b_n S_n$$

$$\Big(\text{weil } \sum_{m=1}^{\infty}b_m=0 \text{ für } n=0\Big)$$

und ebenso ist

$$\sum_{n=0}^{\infty}a_n\sum_{m=1}^{n+1}b_m=\sum_{n=0}^{\infty}b_{n+1}S_n.$$

Wir können folglich (23) und (24) auch in folgender Form schreiben:

$$F,(\tfrac{1}{2}\,\tfrac{1}{2};1;z)=\frac{1}{\pi}log\frac{16}{1-z}\,.\,F(\tfrac{1}{2},\tfrac{1}{2};1;1-z)-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\,2n}\sum_{q=n}^{\infty}\left\{\frac{\Gamma(\frac{1}{2}+q)}{q!}\right\}^2(1-z)^q, \quad (25)$$

$$\left.\begin{aligned}F(-\tfrac{1}{2},\tfrac{1}{2};1;z)=&\frac{2}{\pi}F(-\tfrac{1}{2},\tfrac{1}{2};1;1-z)+\frac{(1-z)}{2\pi}log\frac{16}{1-z}\,.\,F(\tfrac{1}{2},\tfrac{3}{2};2;1-z)\\ &-\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)(2n+2)}\sum_{q=n}^{\infty}\frac{\Gamma(q+\frac{1}{2})\,\Gamma(q+\frac{3}{2})}{q!\,(q+1)!}(1-z)^{q+1}.\end{aligned}\right\} \quad (26)$$

Setzt man

$$l=\sqrt{1-k^2}=\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

und definiert man die doppelte Zahlenfolge $(c_n,\ d_n)$ durch

$$\left.\begin{aligned}c_0=1+l=1+\cos\alpha;\qquad & d_0=1-l=1-\cos\alpha;\\ c_n=\frac{c_{n-1}+d_{n-1}}{2};\qquad & d_n=\sqrt{c_{n-1}\,.\,d_{n-1}};\ (n\geqq 1)\end{aligned}\right\} \quad . \quad (28)$$

so ist für $n\geqq 2$

$$c_n=\frac{c_{n-2}+d_{n-2}+2\sqrt{c_{n-2}\,.\,d_{n-2}}}{4}=\left(\frac{\sqrt{c_{n-2}}+\sqrt{d_{n-2}}}{2}\right)^2 \quad . \quad (29)$$

Aus (28) und (29) ergibt sich

$$\left.\begin{aligned}c_1&=1 && ;d_1=\sin\alpha=k;\\ c_2&=\cos^2\left(\frac{\pi}{4}-\alpha\right) && ;d_2=\sqrt{\sin\alpha}=\sqrt{k};\\ c_3&=\left(\frac{1+\sqrt{\sin\alpha}}{2}\right)^2 && ;d_3=\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).\sqrt[4]{\sin\alpha};\\ c_4&=\left(\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\sqrt[4]{\sin\alpha}}{2}\right)^2 && ;d_4=\sqrt{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).\sqrt[8]{\sin\alpha}\,.\left(\frac{1+\sqrt{\sin\alpha}}{2}\right)};\\ c_5&=\left(\frac{1+\sqrt{\sin\alpha}+2\sqrt{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).\sqrt[8]{\sin\alpha}}}{4}\right)^2; && \\ d_5&=\sqrt[4]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\cdot\sqrt[16]{\sin\alpha}\,.\sqrt{\frac{1+\sqrt{\sin\alpha}}{2}}\,.\,\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\sqrt[4]{\sin\alpha}}{2}; && \end{aligned}\right\} \quad (30)$$

u. s. w.

Weiter definieren wir

$$l_n=\frac{c_{n-1}-d_{n-1}}{c_{n-1}+d_{n-1}}\,(n\geqq 1)=\left(\frac{\sqrt{c_{n-2}}-\sqrt{d_{n-2}}}{\sqrt{c_{n-2}}+\sqrt{d_{n-2}}}\right)^2,\ (n\geqq 2) \quad . \quad (31)$$

also

$$\left.\begin{aligned} l_1 &= l = \cos\alpha = \sqrt{1-k^2};\\ l_2 &= \left(\frac{1-k}{1+k}\right) = tg^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right);\\ l_3 &= \left(\frac{1-\sqrt{k}}{1+\sqrt{k}}\right)^2 = \left(\frac{1-\sqrt{\sin\alpha}}{1+\sqrt{\sin\alpha}}\right)^2;\\ l_4 &= \left(\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-\sqrt[4]{\sin\alpha}}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\sqrt[4]{\sin\alpha}}\right)^2;\\ l_5 &= \left(\frac{1+\sqrt{\sin\alpha}-2\sqrt{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[4]{\sin\alpha}}}{1+\sqrt{\sin\alpha}+2\sqrt{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[4]{\sin\alpha}}}\right)^2; \end{aligned}\right\} \quad . \quad . \quad (32)$$

Die Grössen l_q und k_q sind komplementar, d. h. $l_q(\alpha) = k_q\left(\frac{\pi}{2}-\alpha\right)$.

Weiter ist für $n \geqq 1$

$$\left.\begin{aligned} &1+l_n = \frac{c_{n-1}}{c_n}; \qquad 1-l_n = \frac{d_{n-1}}{c_n};\\ &\sqrt{1-l_n^2} = \frac{d_n}{c_n};\\ &\frac{1-\sqrt{1-l_n^2}}{1+\sqrt{1-l_n^2}} = \frac{c_n-d_n}{c_n+d_n} = l_{n+1};\\ &\frac{1-l_n}{1+l_n} = \frac{d_{n-1}}{c_{n-1}} = \sqrt{1-l_{n-1}^2};\\ &l_{n-1} = \frac{2\sqrt{l_n}}{1+l_n}; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Satz III: Für ganzes $q \geqq 1$ ist

$$\frac{K(k)}{2} = \frac{K(\sqrt{1-l_1^2})}{2\,.\,c_1} = \frac{K(\sqrt{1-l_2^2})}{2^2\,.\,c_2} = \ldots . \frac{K(\sqrt{1-l_q^2})}{2^q\,c_q} = \frac{\pi}{2^{q+1}\,.\,c_q} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; (1-l_q^2)\right) . \quad (34)$$

Beweis: Setzt man in (13) (Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 969 (1941)),

$$p = \sqrt{1-l_n^2} \qquad (n \geqq 1),$$

so ist

$$K(\sqrt{1-l_n^2}) = \frac{1}{1+\sqrt{1-l_n^2}} K\left(\sqrt{1-\left(\frac{1-\sqrt{1-l_n^2}}{1+\sqrt{1-l_n^2}}\right)^2}\right) =$$

$$= \frac{1+l_{n+1}}{2} K\sqrt{1-l_{n+1}^2} = \frac{c_n}{2\,c_{n+1}} K(\sqrt{1-l_{n+1}^2});$$

oder

$$\frac{K(\sqrt{1-l_n^2})}{2^n\, c_n} = \frac{K(\sqrt{1-l_{n+1}^2})}{2^{n+1}\, c_{n+1}} \qquad (n \geqq 1); \qquad \text{w. z. b. w.}$$

Satz IV: Für ganzes $q \geqq 1$ ist

$$K(k) = \frac{1}{2^{q-1} c_q} \log \frac{4}{l_q} \cdot F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; l_q^2\right) - \frac{1}{2^{q-2} c_q \cdot \pi} \cdot \sum_{m=1}^{\infty} \frac{1}{(2m-1)\, 2m} \cdot \sum_{n=m}^{\infty} \left\{\frac{\Gamma(\frac{1}{2}+n)}{n!}\right\}^2 l_q^{2n}. \quad (35)$$

Beweis: Folgt aus (25) und (34).
Aus (35), (30) und (32) findet man für $n = 1, \ldots\ldots, 5$

$$K(k) = \log \frac{4}{\sqrt{1-k^2}} \cdot F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; (1-k^2)\right) - \frac{2}{\pi} \sum_{n=0}^{\infty} \left(\frac{\Gamma(\frac{1}{2}+n)}{n!}\right)^2 (1-k^2)^n \sum_{m=1}^{n} \frac{1}{(2m-1)\, 2m}$$

$$= \log \frac{4}{\sqrt{1-k^2}} \cdot F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; (1-k^2)\right) - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)\, 2m} \sum_{n=m}^{\infty} \left(\frac{\Gamma(\frac{1}{2}+n)}{n!}\right)^2 (1-k^2)^n; \qquad (K^* I)$$

$$K(k) = \frac{1}{\cos^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)} \left\{ \log 2 \cdot \cot g \left(\frac{\pi}{4} - \frac{\alpha}{2}\right) F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; tg^4\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right) \right.$$

$$\left. - \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)\, 2m} \sum_{n=m}^{\infty} \left(\frac{\Gamma(\frac{1}{2}+n)}{n!}\right)^2 tg^{4n}\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \right\}; \quad (K^* II)$$

$$K(k) = \frac{2}{(1+\sqrt{\sin\alpha})^2} \left\{ \log 2 \left(\frac{1+\sqrt{\sin\alpha}}{1-\sqrt{\sin\alpha}}\right) \cdot F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \left(\frac{1-\sqrt{\sin\alpha}}{1+\sqrt{\sin\alpha}}\right)^4\right) \right.$$

$$\left. - \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)\, 2m} \sum_{n=m}^{\infty} \left(\frac{\Gamma(\frac{1}{2}+n)}{n!}\right)^2 \left(\frac{1-\sqrt{\sin\alpha}}{1+\sqrt{\sin\alpha}}\right)^{4n} \right\}; \quad (K^* III)$$

$$K(k) = \frac{1}{\left(\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \sqrt[4]{\sin\alpha}\right)^2} \left\{ \log 2 \left(\frac{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \sqrt[4]{\sin\alpha}}{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) - \sqrt[4]{\sin\alpha}} \right) \cdot \right.$$

$$\left. F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \left(\frac{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) - \sqrt[4]{\sin\alpha}}{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \sqrt[4]{\sin\alpha}} \right)^4 \right) \right)$$

$$- \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)\, 2m} \sum_{n=m}^{\infty} \left(\frac{\Gamma(\frac{1}{2}+n)}{n!}\right)^2 \cdot \left(\frac{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) - \sqrt[4]{\sin\alpha}}{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \sqrt[4]{\sin\alpha}} \right)^{4n} \Bigg\}; \qquad (K^* IV)$$

$$(k)=\frac{2}{\left(1+\sqrt{\sin\alpha}+2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}\right)^2}\cdot$$

$$\left\{\log 2\left(\frac{1+\sqrt{\sin\alpha}+2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}{1+\sqrt{\sin\alpha}-2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}\right)\cdot\right.$$

$$\left.F\left(\tfrac{1}{2},\tfrac{1}{2};1;\left(\frac{1+\sqrt{\sin\alpha}-2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}{1+\sqrt{\sin\alpha}+2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}\right)^4\right)\right)$$

$$\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{(2m-1)\,2m}\sum_{n=m}^{\infty}\left(\frac{\Gamma(\frac{1}{2}+n)}{n!}\right)^2\left(\frac{1+\sqrt{\sin\alpha}-2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}{1+\sqrt{\sin\alpha}+2\sqrt[8]{\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}\right)^{4n}\Bigg\}. \qquad (K^*V)$$

Für Werte von k in der Nähe von $k=1$ ist

$$l_n\approx\frac{\cos^{2^{n-1}}\alpha}{2^{2^n-2}}.$$

Die Konvergenz der Reihen K^* ist somit ausserordentlich stark in der Nähe von $k=1$. Es ist deutlich, dass die Reihen K^* die Gegenstücke zu den gleichnumerierten Reihen K bilden.

Die Hauptglieder der Entwicklungen K^*III—K^*V sind

$$K(k)\approx\frac{2}{(1+\sqrt{k})^2}\log 2\left(\frac{1+\sqrt{k}}{1-\sqrt{k}}\right); \qquad (K^*III')$$

$$K(k)\approx\frac{2}{\left(\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\sqrt[4]{\sin\alpha}\right)^2}\log 2\left(\frac{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)+\sqrt[4]{\sin\alpha}}{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)-\sqrt[4]{\sin\alpha}}\right); \qquad (K^*IV')$$

$$K(k)\approx\frac{2}{\left(1+\sqrt{\sin\alpha}+2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}\right)^2}\times$$

$$\log 2\left(\frac{1+\sqrt{\sin\alpha}+2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}{1+\sqrt{\sin\alpha}-2\sqrt[8]{\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cdot\sqrt[8]{\sin\alpha}}}\right). \qquad (K^*V').$$

Die Fehler für $\frac{\pi}{4}\leqq\alpha\leqq\frac{\pi}{2}$ sind aus der folgenden Tabelle abzulesen.

Für $\alpha = \frac{\pi}{4}$ ist $k_1 = l_1 = \frac{1}{2}\sqrt{2} = 0{,}70710678$.

q	Fehler bei Verwendung nur des Hauptgliedes
3	$\leqq 1{,}9 \cdot 10^{-5}$
4	$7{,}1 \cdot 10^{-11}$
5	$9 \cdot 10^{-21}$

Man ersieht hieraus, dass die Anwendung der einfach gebildeten Hauptglieder

$$\left(K' \text{ für } 0 \leqq \alpha \leqq \frac{\pi}{4};\ K^{*\prime} \text{ für } \frac{\pi}{4} \leqq \alpha \leqq \frac{\pi}{2}\right)$$

schon überall im Gebiet $0 \leqq \alpha \leqq \frac{\pi}{2}$ Genauigkeit verbürgt

in 4 Stellen für $q = 3$,
„ 9 „ „ $q = 4$,
„ 19 „ „ $q = 5$.

Historisch ist noch zu bemerken, dass die Entwicklung (K^*I) sich schon in GAUSS' Nachlass vorfindet[1]), und in Proc. XLIV, S. 606 erwähnt worden ist.

Die Entwicklung (K^*III), deren erste Glieder schon in einer vorhergehenden Arbeit (Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 825 (1941)) in elementarer Weise abgeleitet worden sind, findet sich, soweit mir bekannt ist, zuerst, allerdings ohne Beweis, bei SCHWARZ–WEIERSTRASS, Formeln und Lehrsätze, S. 68, Gl. 10, zweiter Teil.

Als Gegenstück zu Satz II ergibt sich aus (15) und (35)

Satz V: Für ganzes $q \geqq 1$ ist

$$K(k) = \frac{1}{2^{q-1} d_q} \log \frac{4}{l_q} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; -\left(\frac{l_q c_q}{d_q}\right)^2\right) - \frac{1}{2^{q-2} c_q} \cdot \pi \sum_{m=1}^{\infty} \frac{1}{(2m-1)\,2m} \cdot \sum_{n=m}^{\infty} \left\{\frac{\Gamma\left(\frac{1}{2}+n\right)}{n!}\right\}^2 l_q^{2n}. \quad (36)$$

Für $q = 3, 4, 5$ findet man die Hauptglieder

$$K(k) \approx \frac{1}{2\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \sqrt[4]{\sin\alpha}} \log 2 \left(\frac{1 + \sqrt{\sin\alpha}}{1 - \sqrt{\sin\alpha}}\right); \quad (K.K^* III')$$

$$K(k) \approx \frac{1}{2\sqrt{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \cdot \sqrt[8]{\sin\alpha} \cdot (1 + \sqrt{\sin\alpha})}} \log 2 \left(\frac{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \sqrt[4]{\sin\alpha}}{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) - \sqrt[4]{\sin\alpha}}\right); \quad (K.K^* IV')$$

$$K(k) \approx \frac{1}{4\sqrt[4]{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)} \cdot \sqrt[16]{\sin\alpha} \cdot \sqrt{\frac{1 + \sqrt{\sin\alpha}}{2}} \cdot \left(\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) + \sqrt[4]{\sin\alpha}\right)} \times$$

$$\log 2 \left(\frac{1 + \sqrt{\sin\alpha} + 2\sqrt{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \cdot \sqrt[8]{\sin\alpha}}}{1 + \sqrt{\sin\alpha} - 2\sqrt{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \cdot \sqrt[8]{\sin\alpha}}}\right) \quad (K.K^* V').$$

(Fortsetzung folgt.)

[1]) GAUSS, Werke III, S. 217, Gl. [90].

Mathematics. — *Ueber die Entwicklung der unvollständigen elliptischen Integrale erster und zweiter Art in stark konvergenten Reihen.* II. Von S. C. VAN VEEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 25, 1941.)

§ 1. *β beliebig; α klein* $\left(\text{nicht in der Nähe von } \frac{\pi}{2}\right)$.

A. Das unvollständige elliptische Integral erster Art.

Wir betrachten das u. e. I. erster Art

$$F(k_n, \beta_n) = \int_0^{\beta_n} \frac{d\varphi_n}{\sqrt{1-k_n^2 \sin^2 \varphi_n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Unter Anwendung der LANDENschen Transformation

$$tg\, \varphi_n = \frac{\sin 2\varphi_{n-1}}{k_n + \cos 2\varphi_{n-1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

geht (1) bekanntlich über in [1])

$$\left.\begin{aligned} \frac{2}{1+k_n} \int_0^{\beta_{n-1}} \frac{d\varphi_{n-1}}{\sqrt{1-\frac{4k_n}{(1+k_n)^2}\sin^2\varphi_{n-1}}} &= \frac{2}{1+k_n} F(k_{n-1}, \beta_{n-1}) \\ \text{mit} \qquad k_{n-1} = \frac{2\sqrt{k_n}}{1+k_n};\ tg\, \beta_n &= \frac{\sin 2\beta_{n-1}}{k_n + \cos 2\beta_{n-1}};\ (n \geqq 2). \end{aligned}\right\} \quad . \quad . \quad (3)$$

Unter Anwendung der Hilfsgrössen a_n, b_n aus V. E. I; I [2]) ergibt sich aus (1) und (3)

$$\frac{1}{2^n a_n} F(k_n, \beta_n) = \frac{1}{2^{n-1} a_{n-1}} F(k_{n-1}, \beta_{n-1});\ (n \text{ ganz } \geqq 2). \quad . \quad . \quad (4)$$

Aus (3) ergibt sich

$$\left.\begin{aligned} e^{2i\beta_n} &= \frac{k_n + e^{2ib_{n-1}}}{k_n + e^{-2i\beta_{n-1}}} (n \geqq 2) = \frac{k_n + \dfrac{k_{n-1} + e^{2i\beta_{n-2}}}{k_{n-1} + e^{-2i\beta_{n-2}}}}{k_n + \dfrac{k_{n-1} + e^{-2i\beta_{n-2}}}{k_{n-1} + e^{2i\beta_{n-2}}}} \\ &= \frac{k_{n-1} + e^{2i\beta_{n-2}}}{k_{n-1} + e^{-2i\beta_{n-2}}} \cdot \frac{k_n . k_{n-1} + k_{n-1} + k_n e^{-2i\beta_{n-2}} + e^{2i\beta_{n-2}}}{k_n . k_{n-1} + k_{n-1} + k_n e^{2i\beta_{n-2}} + e^{-2i\beta_{n-2}}} = \\ &= e^{4i\beta_{n-2}} \cdot \frac{k_{n-1} + e^{2i\beta_{n-2}}}{k_{n-1} + e^{-2i\beta_{n-2}}} \cdot \frac{1 + 2\sqrt{k_n}\, e^{-2i\beta_{n-2}} + k_n e^{-4i\beta_{n-2}}}{1 + 2\sqrt{k_n}\, e^{2i\beta_{n-2}} + k_n e^{4i\beta_{n-2}}} \left(\text{wegen } k_{n-1} = \frac{2\sqrt{k_n}}{1+k_n}\right) \\ &= e^{4i\beta_{n-2}} \cdot \frac{k_{n-1} + e^{2i\beta_{n-2}}}{k_{n-1} + e^{-2i\beta_{n-2}}} \cdot \left(\frac{1 + \sqrt{k_n} . e^{-2i\beta_{n-2}}}{1 + \sqrt{k_n} . e^{2i\beta_{n-2}}}\right)^2 = \\ &= e^{2i\beta_{n-2}} \cdot \frac{1 + k_{n-1} . e^{-2i\beta_{n-2}}}{k_{n-1} + e^{-2i\beta_{n-2}}} \cdot \left(\frac{1 + \sqrt{k_n} . e^{-2i\beta_{n-2}}}{\sqrt{k_n} + e^{-2i\beta_{n-2}}}\right)^2;\quad (n \geqq 3). \end{aligned}\right\} \quad (5)$$

[1]) Vgl. Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 607 (1941), Hilfssatz 1.

[2]) Abkürzung für „Stark konvergente Entwicklungen für die vollständigen elliptischen Integrale erster und zweiter Art, I", (Proc. Ned. Akad. v. Wetensch., Amsterdam, **44**, 964—973 (1941).

Wegen [1])

$$k_n = \frac{a_{n-1} - b_{n-1}}{a_{n-1} + b_{n-1}} = \left(\frac{\sqrt{a_{n-2}} - \sqrt{b_{n-2}}}{\sqrt{a_{n-2}} + \sqrt{b_{n-2}}}\right)^2 \qquad (n \geqq 2)$$

ist

$$\left.\begin{aligned} &\frac{1 + k_{n-1} \cdot e^{-2i\beta_{n-2}}}{k_{n-1} + e^{-2i\beta_{n-2}}} = \frac{a_{n-2}(1 + e^{-2i\beta_{n-2}}) + b_{n-2}(1 - e^{-2i\beta_{n-2}})}{a_{n-2}(1 + e^{-2i\beta_{n-2}}) - b_{n-2}(1 - e^{-2i\beta_{n-2}})} = \\ &= \frac{1 + i\frac{b_{n-2}}{a_{n-2}} \cdot tg\,\beta_{n-2}}{1 - i\frac{b_{n-2}}{a_{n-2}} \cdot tg\,\beta_{n-2}}; \qquad (n \geqq 3) \end{aligned}\right\} \quad (6)$$

und

$$\left.\begin{aligned} &\frac{1 + \sqrt{k_n}\, e^{-2i\beta_{n-2}}}{\sqrt{k_n} + e^{-2i\beta_{n-2}}} = \frac{\sqrt{a_{n-2}}(1 + e^{-2i\beta_{n-2}}) + \sqrt{b_{n-2}}(1 - e^{-2i\beta_{n-2}})}{\sqrt{a_{n-2}}(1 + e^{-2i\beta_{n-2}}) - \sqrt{b_{n-2}}(1 - e^{-2i\beta_{n-2}})} = \\ &= \frac{1 + i\sqrt{\frac{b_{n-2}}{a_{n-2}}} \cdot tg\,\beta_{n-2}}{1 - i\sqrt{\frac{b_{n-2}}{a_{n-2}}} \cdot tg\,\beta_{n-2}}; \qquad (n \geqq 3) \end{aligned}\right\} \quad (7)$$

Also ergibt sich aus (5), (6) und (7)

$$e^{2i\beta_n} = e^{2i\beta_{n-2}} \left(\frac{1 + i\frac{b_{n-2}}{a_{n-2}} \cdot tg\,\beta_{n-2}}{1 - i\frac{b_{n-2}}{a_{n-2}} \cdot tg\,\beta_{n-2}}\right) \cdot \left(\frac{1 + i\sqrt{\frac{b_{n-2}}{a_{n-2}}} \cdot tg\,\beta_{n-2}}{1 - i\sqrt{\frac{b_{n-2}}{a_{n-2}}} \cdot tg\,\beta_{n-2}}\right)^2 \quad . \quad . \quad (8)$$

$(n \geqq 3)$

und für $n = 3$, nach (V.E. I; I) Formel 8

$$\left.\begin{aligned} e^{2i\beta_3} &= e^{2i\beta_1} \cdot \left(\frac{1 + i\sqrt{1 - k_1^2} \cdot tg\,\beta_1}{1 - i\sqrt{1 - k_1^2} \cdot tg\,\beta_1}\right) \cdot \left(\frac{1 + i\sqrt{1 - k_1^2} \cdot tg\,\beta_1}{1 - i\sqrt{1 - k_1^2} \cdot tg\,\beta_1}\right)^2 = \\ &= e^{2i\beta_1} \left(\frac{1 + i\cos\alpha \cdot tg\,\beta_1}{1 - i\cos\alpha \cdot tg\,\beta_1}\right) \cdot \left(\frac{1 + i\sqrt{\cos\alpha} \cdot tg\,\beta_1}{1 - i\sqrt{\cos\alpha} \cdot tg\,\beta_1}\right)^2 \end{aligned}\right\} \quad (9)$$

oder

$$\beta_3 = \beta_1 + arc \cdot tg\,(\cos\alpha \cdot tg\,\beta_1) + 2\,arc \cdot tg\,(\sqrt{\cos\alpha} \cdot tg\,\beta_1). \quad . \quad (10)$$

Schliesslich ergibt sich aus (4), (nach V. E. I; I; Formel (3)).

$$F(k_1, \beta_1) = \frac{1}{4\,a_3} F(k_3, \beta_3),$$

---

[1]) V. E. I; I, Formel (4).

oder

$$\int_0^{\beta_1} \frac{d\varphi}{\sqrt{1-\sin^2\alpha\,.\,\sin^2\varphi}} = \frac{1}{(1+\sqrt{\cos\alpha})^2} \int_0^{\beta_3} \frac{d\varphi}{\sqrt{1-\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4 \sin^2\varphi}} =$$

$$= \frac{1}{(1+\sqrt{\cos\alpha})^2} \sum_{n=0}^{\infty} \left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right) \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n} \cdot \int_0^{\beta_3} \sin^{2n}\varphi\, d\varphi.$$

Weiter ist

$$\int_0^{\beta_3} \sin^{2n}\varphi\, d\varphi = \frac{(-1)^n}{2^{2n}} \int_0^{\beta_3} (e^{i\varphi}-e^{-i\varphi})^{2n}\, d\varphi = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \int_0^{\beta_3} e^{2(n-k)i\varphi}\, d\varphi =$$

$$= \left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right)\beta_3 + \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cdot \frac{\sin 2(n-k)\beta_3}{n-k}, \quad (n \geqq 1)$$

also, wegen (10)

$$\left.\begin{aligned} F(k_1,\beta_1) = {} & \frac{1}{(1+\sqrt{\cos\alpha})^2} (\beta_1 + arc\,.\,tg(\cos\alpha\,.\,tg\,\beta_1) + 2\,arc\,.\,tg(\sqrt{\cos\alpha}\,.\,tg\,\beta_1)) \\ & \sum_{n=0}^{\infty} \left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right)^2 \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n} + \\ + \frac{1}{(1+\sqrt{\cos\alpha})^2} & \sum_{n=1}^{\infty} \left(\frac{1.3.5\ldots(2n-1)}{2..6\ldots 2n}\right) \frac{(-1)^n}{2^{2n}} \cdot \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n} \cdot \\ & \cdot \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cdot \frac{\sin 2(n-k)\beta_3}{n-k}. \end{aligned}\right\} \quad (11)$$

wo β_3 durch (10) bestimmt ist.

Die Reihe (11) konvergiert sehr stark, wenn α nicht in der Nähe von $\frac{\pi}{2}$ liegt.

Für $\alpha = \frac{\pi}{2}$ hört die Konvergenz auf. Aber selbst wenn α ziemlich dicht bei $\frac{\pi}{2}$ liegt, ist die Konvergenz hinreichend, z. B. für $\alpha \leqq 70°$ ist schon

$$k = \sin\alpha \leqq 0{,}94, \text{ und } \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4 < 0{,}005.$$

Für kleinere Werte von α ist die Konvergenz dermassen stark, dass man sich in weitaus den meisten praktischen Rechnungen mit dem Hauptgliede

$$F(k_1,\beta_1) \approx \frac{1}{(1+\sqrt{\cos\alpha})^2} (\beta_1 + arc\,.\,tg(\cos\alpha\,.\,tg\,\beta_1) + 2\,arc\,.\,tg(\sqrt{\cos\alpha}\,.\,tg\,\beta_1)) \quad (12)$$

begnügen kann.

Für $\beta = \frac{\pi}{2}$ geht (11) in

$$K(\sin\alpha) = \frac{2\pi}{(1+\sqrt{\cos\alpha})^2} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\right) \quad (13)$$

über. (Vgl. V. E. I; I, Formel *K III*).

Ohne Beschränkung der Allgemeinheit kann in (11)

$$\beta_1 \leqq \frac{\pi}{2}, \text{ also } \beta_3 \leqq 2\pi, \text{ (wegen (10))}$$

genommen werden, denn für

$$\beta_1 = m\,\frac{\pi}{2} + \gamma, \quad \left(m \text{ ganz} \geqq 1;\ 0 \leqq \gamma < \frac{\pi}{2}\right)$$

ist

$$F(k_1, \beta_1) = m\,.\,K(\sin\alpha) + F(k_1, \gamma) \quad \text{oder} \quad (m+1)\,.\,K(\sin\alpha) - F\left(k_1, \frac{\pi}{2} - \gamma\right)$$

je nachdem m gerade oder ungerade ist.

Für $\alpha \leqq \frac{\pi}{4}$, $\beta_1 \leqq \frac{\pi}{2}$ ist der Fehler bei Verwendung des Hauptgliedes (12) $< 3.10^{-5}$.

B. Das unvollständige elliptische Integral zweiter Art.
Wir betrachten das u. e. I. zweiter Art

$$E(k_n, \beta_n) = \int_0^{\beta_n} \sqrt{1 - k_n^2 \sin^2\varphi_n}\, d\varphi_n \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Unter Anwendung der LANDENschen Transformation

$$tg\,\varphi_n = \frac{\sin 2\varphi_{n-1}}{k_n + \cos 2\varphi_{n-1}}$$

findet man

$$E(k_n, \beta_n) + k_n \sin\beta_n = \int_0^{\beta_n} (\sqrt{1 - k_n^2 \sin^2\varphi_n} + k_n \cos\varphi_n)\, d\varphi_n =$$

$$= 2\int_0^{\beta_{n-1}} \frac{1 + k_n \cos 2\varphi_{n-1}}{\sqrt{1 + 2k_n \cos 2\varphi_{n-1} + k_n^2}}\, d\varphi_{n-1} = \frac{2}{1+k_n} \int_0^{\beta_{n-1}} \frac{1 + k_n - 2k_n \sin^2\varphi_{n-1}}{\sqrt{1 - \frac{4k_n}{(1+k_n)^2}\sin^2\varphi_{n-1}}}\, d\varphi_{n-1}$$

$$= (1+k_n) \int_0^{\beta_{n-1}} \sqrt{1 - k_{n-1}^2 \sin^2\varphi_{n-1}}\, d\varphi_{n-1} + (1-k_n) \int_0^{\beta_{n-1}} \frac{d\varphi_{n-1}}{\sqrt{1 - k_{n-1}^2 \sin^2\varphi_{n-1}}},$$

also, nach V. E. I; I, Formel (6) und (7)

$$E(k_n, \beta_n) + k_n \sin\beta_n = \frac{a_{n-1}}{a_n} E(k_{n-1}, \beta_{n-1}) + \frac{b_{n-1}}{a_n} F(k_{n-1}, \beta_{n-1}); \quad (15)$$

oder

$$a_n E(k_n, \beta_n) - a_{n-1} E(k_{n-1}, \beta_{n-1}) = b_{n-1} F(k_{n-1}, \beta_{n-1}) - a_n k_n \sin\beta_n; \ (n \geqq 2).$$

Insbesondere ist

$$\left.\begin{aligned} a_3 E(k_3, \beta_3) - a_1 E(k_1, \beta_1) &= b_2 F(k_2, b_2) + b_1 F(k_1, \beta_1) - a_3 k_3 \sin\beta_3 - a_2 k_2 \sin\beta_2 = \\ &= \frac{a_1 b_1 + 2a_2 b_2}{4a_3} F(k_3, b_3) - a_3 k_3 \sin\beta_3 - a_2 k_2 \sin\beta_2, \end{aligned}\right\} \quad (16)$$

wegen (4)

$$e^{2i\beta_2} = e^{2i\beta_1} \cdot \frac{1 + k_2 e^{-2i\beta_1}}{k_2 + e^{-2i\beta_1}} = e^{2i\beta_1} \cdot \frac{1 + i\frac{b_1}{a_1} tg\,\beta_1}{1 - i\frac{b_1}{a_1} tg\,\beta_1}$$

also

$$\beta_2 = \beta_1 + arc\,tg\,(\cos\alpha \cdot tg\,\beta_1).$$

Weiter ist

$$e^{2i\beta_2} = \frac{e^{2i\beta_1}\left(1 + i\frac{b_1}{a_1} tg\,\beta_1\right)^2}{1 + \frac{b_1^2}{a_1^2} tg^2\,\beta_1} = \frac{e^{2i\beta_1}(1 + i\cos\alpha \cdot tg\,\beta_1)^2}{1 + \cos^2\alpha \cdot tg^2\,\beta_1}$$

oder

$$e^{i\beta_2} = \frac{e^{i\beta_1}(1 + i\cos\alpha \cdot tg\,\beta_1)\cos\beta_1}{\triangle}$$

mit

$$\cos\beta_1 \sqrt{1 + \cos^2\alpha\, tg^2\,\beta_1} = \sqrt{1 - \sin^2\alpha \cdot \sin^2\beta_1} = \triangle. \quad . \quad (17)$$

Durch Trennung der reellen und imaginären Teile findet man

$$\cos\beta_2 = \frac{\cos^2\beta_1 - \sin\beta_1\cos\beta_1\cos\alpha\, tg\,\beta_1}{\triangle} = \frac{\cos^2\beta_1(1 - \cos\alpha\, tg^2\,\beta_1)}{\triangle} \quad . \quad (18)$$

$$\sin\beta_2 = \frac{\sin\beta_1\cos\beta_1 + \cos^2\beta_1\cos\alpha\, tg\,\beta_1}{\triangle} = \frac{\sin 2\beta_1 \cos^2\frac{\alpha}{2}}{\triangle} \quad . \quad . \quad (19)$$

Wegen (9) ist:

$$e^{i\beta_3} = \frac{e^{i\beta_1}(1 + i\cos\alpha \cdot tg\,\beta_1)}{\sqrt{1 + \cos^2\alpha \cdot tg^2\,\beta_1}} \frac{(1 + i\sqrt{\cos\alpha} \cdot tg\,\beta_1)}{(1 - i\sqrt{\cos\alpha}\, tg\,\beta_1)} =$$

$$= \frac{e^{i\beta_1}(1 + i\cos\alpha \cdot tg\,\beta_1)\,(1 + i\sqrt{\cos\alpha} \cdot tg\,\beta_1)^2\cos\beta_1}{\triangle\,(1 + \cos\alpha \cdot tg^2\,\beta_1)},$$

also

$$\sin\beta_3 = \frac{\sin\beta_1\cos\beta_1(1 - \cos\alpha \cdot tg^2\,\beta_1)(1 + \sqrt{\cos\alpha})^2}{\triangle \cdot (1 + \cos\alpha \cdot tg^2 \cdot \beta_1)} \quad . \quad . \quad . \quad (20)$$

Wegen V. E. I; I, Formel (3) und (5) ist

$$\left.\begin{aligned} a_3 k_3 &= \tfrac{1}{4}(1 - \sqrt{\cos\alpha})^2 \\ a_2 k_2 &= \sin^2\frac{\alpha}{2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

und, wegen (19), (20) und (21)

$$\left.\begin{aligned} a_2 k_2 \sin\beta_2 + a_3 k_3 \sin\beta_3 &= \frac{\sin 2\beta_1 \cdot \sin^2\alpha}{4\cdot\triangle} + \frac{\sin 2\beta_1 \sin^4\frac{\alpha}{2}\,(1-\cos\alpha\cdot tg^2\beta_1)}{2\triangle(1+\cos\alpha\cdot tg^2\beta_1)} = \\ &= \frac{\sin 2\beta_1\cdot\sin^2\frac{\alpha}{2}}{2\triangle(1+\cos\alpha\cdot tg^2\beta_1)}\cdot\left(2\cos^2\frac{\alpha}{2}\,(1+\cos\alpha\cdot tg^2\beta_1)+\sin^2\frac{\alpha}{2}\,(1-\cos\alpha\cdot tg^2\beta_1)\right) = \\ &= \frac{\sin 2\beta_1\cdot\sin^2\frac{\alpha}{2}}{2\triangle(1+\cos\alpha\cdot tg^2\beta_1)}\left(\cos^2\frac{\alpha}{2}\,(1+\cos\alpha\cdot tg^2\beta_1)+\frac{\triangle^2}{\cos^2\beta_1}\right) = \\ &= \frac{\sin 2\beta_1\sin^2\alpha}{8\triangle} + \frac{tg\,\beta_1\cdot\sin^2\frac{\alpha}{2}\cdot\triangle}{1+\cos\alpha\cdot tg^2\beta_1} \end{aligned}\right\} \quad (22)$$

Schliesslich ergibt sich aus (16) und (22)

$$E(k_1,\beta_1) = \frac{\sin 2\beta\cdot\sin^2\alpha}{8\triangle} + \frac{tg\,\beta_1\cdot\sin^2\frac{\alpha}{2}\cdot\triangle}{1+\cos\alpha\cdot tg^2\beta_1} + \left(\frac{1+\sqrt{\cos\alpha}}{2}\right)^2\int_0^{\beta_3}\sqrt{1-\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\sin^2\varphi}\,d\varphi$$

$$-\sqrt{\cos\alpha}\,\frac{(1+\sqrt{\cos\alpha}+\cos\alpha)}{(1+\sqrt{\cos\alpha})^2}\int_0^{\beta_3}\frac{d\varphi}{\sqrt{1-\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^4\sin^2\varphi}} =$$

$$= \frac{\sin 2\beta_1\cdot\sin^2\alpha}{8\triangle} + \frac{tg\,\beta_1\cdot\sin^2\frac{\alpha}{2}\cdot\triangle}{1+\cos\alpha\,tg^2\beta_1} +$$

$$\left(\frac{1+\sqrt{\cos\alpha}}{2}\right)^2\sum_{n=0}^{\infty}\left(\frac{-1\cdot1\cdot3\ldots(2n-3)}{2\cdot4\cdot6\ldots2n}\right)\cdot\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n}\int_0^{\beta_3}\sin^{2n}\varphi\,d\varphi$$

$$-\sqrt{\cos\alpha}\,\frac{(1+\sqrt{\cos\alpha}+\cos\alpha)}{(1+\sqrt{\cos\alpha})^2}\sum_{n=0}^{\infty}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n}\cdot\int_0^{\beta_3}\sin^{2n}\varphi\cdot d\varphi =$$

$$= \frac{\sin 2\beta_1\cdot\sin^2\alpha}{8\triangle} + \frac{tg\,\beta_1\cdot\sin^2\frac{\alpha}{2}\cdot\triangle}{1+\cos\alpha\cdot tg^2\beta_1} -$$

$$\left(\frac{1+\sqrt{\cos\alpha}}{2}\right)^2\beta_3\sum_{n=0}^{\infty}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2\frac{1}{2n-1}\cdot\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n}$$

$$-\sqrt{\cos\alpha}\,\frac{(1+\sqrt{\cos\alpha}+\cos\alpha)}{(1+\sqrt{\cos\alpha})^2}\cdot\beta_3\cdot\sum_{n=0}^{\infty}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n} +$$

$$+\left(\frac{1+\sqrt{\cos\alpha}}{2}\right)^2 \sum_{n=1}^{\infty}\left(\frac{-1.1.3\ldots(2n-3)}{2.4.6\ldots 2n}\right)\frac{(-1)^n}{2^{2n}}$$

$$\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\frac{\sin 2(n-k)\beta_3}{n-k}$$

$$-\sqrt{\cos\alpha}\,\frac{(1+\sqrt{\cos\alpha}+\cos\alpha)}{(1+\sqrt{\cos\alpha})^2}\cdot\sum_{n=1}^{\infty}\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\right)\frac{(-1)^n}{2^{2n}}$$

$$\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right)^{4n}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\frac{\sin 2(n-k)\beta_3}{n-k}. \quad (23)$$

Das Hauptglied

$$\left.\begin{aligned} E(k_1,\beta_1) \approx \frac{\sin 2\beta\,.\,\sin^2\alpha}{8\triangle} + \frac{tg\,\beta_1\,.\,\sin^2\frac{\alpha}{2}\,.\,\triangle}{1+\cos\alpha\,.\,tg^2\beta_1} + \\ \frac{\cos^4\frac{\alpha}{2}}{(1+\sqrt{\cos\alpha})^2}(\beta_1 + arc\,.\,tg\,.\,(\cos\alpha\,.\,tg\,\beta_1) + 2\,arc\,.\,tg\,(\sqrt{\cos\alpha}\,.\,tg\,\beta_1)) \end{aligned}\right\} \quad (24)$$

wird für kleines α in den meisten praktischen Rechnungen genügen.

(Fortsetzung folgt.)

Mathematics. — *Sur des séries et des intégrales définies contenantes les fonctions de* BESSEL. VI. By J. G. RUTGERS. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of October 25, 1941.)

§ 14. Remplaçons dans (296) à cause de $I\ \varrho$ par $\varrho + m + \frac{1}{2}$ et multiplions les deux membres par $(-1)^m (2\varrho + 2m + 1)$, alors nous trouvons après la sommation sur m de 0 à ∞ et en appliquant le théorème de CAUCHY (g):

$$\left.\begin{aligned} &\sum_{s=0}^{\infty} (-1)^s (s+1)(2\varrho+s+1)\, I_{\nu+\varrho+s+1}(x)\, I_{\nu+\varrho+s+\frac{1}{2}}(x) = \\ &= \frac{1}{\sqrt{x}} \int_0^x I_\nu(x-a)\, I_{\nu-\frac{1}{2}}(x-a)\, I_{2\varrho}(2a)\, a \sqrt{x-a}\, da, \quad \begin{cases} R(\nu) > -\frac{1}{2} \\ R(\varrho) > -\frac{1}{2} \end{cases} \end{aligned}\right\} \quad (377)$$

ou en posant $\nu = \mu - \varrho - \frac{1}{2}$;

$$\left.\begin{aligned} &\sum_{s=0}^{\infty} (-1)^s (s+1)(2\varrho+s+1)\, I_{\mu+s+\frac{1}{2}}(x)\, I_{\mu+s}(x) = \\ &= \frac{1}{\sqrt{x}} \int_0^x I_{\mu-\varrho-\frac{1}{2}}(x-a)\, I_{\mu-\varrho-1}(x-a)\, I_{2\varrho}(2a)\, a \sqrt{x-a}\, da, \quad R(\mu) > R(\varrho) > -\tfrac{1}{2}. \end{aligned}\right\} \quad (378)$$

La substitution $\nu = 0$ et $\nu = \frac{1}{2}$ dans (377) donne les formules particulières:

$$\left.\begin{aligned} &\sum_{s=0}^{\infty} (-1)^s (s+1)(2\varrho+s+1)\, I_{\varrho+s+1}(x)\, I_{\varrho+s+\frac{1}{2}}(x) = \\ &= \sqrt{\frac{2}{\pi x}} \int_0^x I_0(x-a)\, I_{2\varrho}(2a) \cos(x-a) \,.\, a\, da, \quad R(\varrho) > -\tfrac{1}{2} \end{aligned}\right\} \quad (379)$$

$$\left.\begin{aligned} &\sum_{s=0}^{\infty} (-1)^s (s+1)(2\varrho+s+1)\, I_{\varrho+s+\frac{3}{2}}(x)\, I_{\varrho+s+1}(x) = \\ &= \sqrt{\frac{2}{\pi x}} \int_0^x I_0(x-a)\, I_{2\varrho}(2a) \sin(x-a) \,.\, a\, da, \quad R(\varrho) > -\tfrac{1}{2}. \end{aligned}\right\} \quad (380)$$

En multipliant les deux membres de (379) par $\sin x$ resp. $\cos x$ et ceux de (380) par $\cos x$ resp. $\sin x$, on trouve après la soustraction resp. l'addition des membres correspondants:

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)(2\varrho+s+1)I_{\varrho+s+1}(x)\{\sin x\,I_{\varrho+s+\frac{1}{2}}(x)-\cos x\,I_{\varrho+s+\frac{3}{2}}(x)\}=\\ =\sqrt{\frac{2}{\pi x}}\int_0^x I_0(x-\alpha)\,I_{2\varrho}(2\alpha)\sin\alpha\,.\,\alpha\,d\alpha,\quad R(\varrho)>-\tfrac{1}{2}\end{aligned}\right\}\quad(381)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)(2\varrho+s+1)I_{\varrho+s+1}(x)\{\cos x\,I_{\varrho+s+\frac{1}{2}}(x)+\sin x\,I_{\varrho+s+\frac{3}{2}}(x)\}=\\ =\sqrt{\frac{2}{\pi x}}\int_0^x I_0(x-\alpha)\,I_{2\varrho}(2\alpha)\cos\alpha\,.\,\alpha\,d\alpha,\quad R(\varrho)>-\tfrac{1}{2}.\end{aligned}\right\}\quad(382)$$

La substitution $\varrho=-\frac{1}{4}$ et $\varrho=\frac{1}{4}$ dans (378) jusqu'à (382) donne les formules particulières:

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+1)I_{\mu+s+\frac{1}{2}}(x)\,I_{\mu+s}(x)=\\ =\frac{2}{\sqrt{\pi x}}\int_0^x I_{\mu-\frac{1}{4}}(\alpha)\,I_{\mu-\frac{1}{4}}(\alpha)\cos 2(x-\alpha)\,.\,\sqrt{\alpha(x-\alpha)}\,d\alpha,\quad R(\mu)>-\tfrac{1}{4}\end{aligned}\right\}\quad(383)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+3)I_{\mu+s+\frac{1}{2}}(x)\,I_{\mu+s}(x)=\\ =\frac{2}{\sqrt{\pi x}}\int_0^x I_{\mu-\frac{3}{4}}(\alpha)\,I_{\mu-\frac{1}{4}}(\alpha)\sin 2(x-\alpha)\,.\,\sqrt{\alpha(x-\alpha)}\,d\alpha,\quad R(\mu)>\tfrac{1}{4}\end{aligned}\right\}\quad(384)$$

ou en remplaçant μ par $\mu+\frac{1}{2}$:

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+3)I_{\mu+s+1}(x)\,I_{\mu+s+\frac{1}{2}}(x)=\\ =\frac{2}{\sqrt{\pi x}}\int_0^x I_{\mu-\frac{1}{4}}(\alpha)\,I_{\mu-\frac{3}{4}}(\alpha)\sin 2(x-\alpha)\,.\,\sqrt{\alpha(x-\alpha)}\,d\alpha,\quad R(\mu)>-\tfrac{1}{2}\end{aligned}\right\}\quad(384')$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+1)I_{s+\frac{3}{4}}(x)\,I_{s+\frac{1}{4}}(x)=\\ =\frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\cos(x-\alpha)\cos 2\alpha\,.\,\sqrt{\alpha}\,d\alpha,\end{aligned}\right\}\quad..\quad(385)$$

$$\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+3)I_{s+\frac{3}{4}}(x)\,I_{s+\frac{1}{4}}(x)= \left.\begin{aligned} &= \frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\cos(x-\alpha)\sin 2\alpha\,.\sqrt{\alpha}\,d\alpha, \end{aligned}\right\} \quad . \quad . \quad (386)$$

$$\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+1)I_{s+\frac{3}{4}}(x)\,I_{s+\frac{1}{4}}(x)= \left.\begin{aligned} &= \frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\sin(x-\alpha)\cos 2\alpha\,.\sqrt{\alpha}\,d\alpha, \end{aligned}\right\} \quad . \quad . \quad (387)$$

$$\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+3)I_{s+\frac{5}{4}}(x)\,I_{s+\frac{3}{4}}(x)= \left.\begin{aligned} &= \frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\sin(x-\alpha)\sin 2\alpha\,.\sqrt{\alpha}\,d\alpha, \end{aligned}\right\} \quad . \quad . \quad (388)$$

$$\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+1)I_{s+\frac{1}{4}}(x)\{\sin x\,I_{s+\frac{3}{4}}(x)-\cos x\,I_{s+\frac{5}{4}}(x)\}= \left.\begin{aligned} &= \frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\sin\alpha\cos 2\alpha\,.\sqrt{\alpha}\,d\alpha, \end{aligned}\right\} \quad (389)$$

$$\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+3)I_{s+\frac{5}{4}}(x)\{\sin x\,I_{s+\frac{3}{4}}(x)-\cos x\,I_{s+\frac{7}{4}}(x)\}= \left.\begin{aligned} &= \frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\sin\alpha\sin 2\alpha\,.\sqrt{\alpha}\,d\alpha, \end{aligned}\right\} \quad (390)$$

$$\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+1)I_{s+\frac{3}{4}}(x)\{\cos x\,I_{s+\frac{1}{4}}(x)+\sin x\,I_{s+\frac{5}{4}}(x)\}= \left.\begin{aligned} &= \frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\cos\alpha\cos 2\alpha\,.\sqrt{\alpha}\,d\alpha, \end{aligned}\right\} \quad (391)$$

$$\sum_{s=0}^{\infty}(-1)^s(s+1)(2s+3)I_{s+\frac{5}{4}}(x)\{\cos x\,I_{s+\frac{3}{4}}(x)+\sin x\,I_{s+\frac{7}{4}}(x)\}= \left.\begin{aligned} &= \frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(x-\alpha)\cos\alpha\sin 2\alpha\,.\sqrt{\alpha}\,d\alpha. \end{aligned}\right\} \quad (392)$$

L'addition et la soustraction de (385) et (388), ainsi de (386) et (387), de (389) et (392), de (390) et (391) donnent:

$$\sum_{s=0}^{\infty}(-1)^s(2s+1)\,I_{s+\frac{3}{4}}(x)\,I_{s+\frac{1}{4}}(x)=\frac{2}{\pi}\Big|\,\frac{2}{x}\int_0^x I_0(x-\alpha)\cos(3\alpha-x)\,.\sqrt{\alpha}\,d\alpha, \tag{393}$$

$$\sum_{s=0}^{\infty}(-1)^s(2s+1)^2\,I_{s+\frac{3}{4}}(x)\,I_{s+\frac{1}{4}}(x)=\frac{2}{\pi}\Big|\,\frac{2}{x}\int_0^x I_0(x-\alpha)\cos(x+\alpha)\,.\sqrt{\alpha}\,d\alpha, \tag{394}$$

$$\sum_{s=0}^{\infty}(-1)^s(2s+2)^2\,I_{s+\frac{1}{4}}(x)\,I_{s+\frac{3}{4}}(x)=\frac{2}{\pi}\Big|\,\frac{2}{x}\int_0^x I_0(x-\alpha)\sin(x+\alpha)\,.\sqrt{\alpha}\,d\alpha, \tag{395}$$

$$\sum_{s=0}^{\infty}(-1)^s(2s+2)\,I_{s+\frac{1}{4}}(x)\,I_{s+\frac{3}{4}}(x)=\frac{2}{\pi}\Big|\,\frac{2}{x}\int_0^x I_0(x-\alpha)\sin(3\alpha-x)\,.\sqrt{\alpha}\,d\alpha, \tag{396}$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s\,I_{s+\frac{3}{4}}(x)\{(2s+1)\sin x\,I_{s+\frac{1}{4}}(x)+(2s+2)\cos x\,I_{s+\frac{5}{4}}(x)\}=\\ =\frac{2}{\pi}\Big|\frac{2}{x}\int_0^x I_0(x-\alpha)\sin 3\alpha\,.\sqrt{\alpha}\,d\alpha,\end{aligned}\right\} \tag{397}$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s\,I_{s+\frac{3}{4}}(x)\{(2s+2)^2\cos x\,I_{s+\frac{5}{4}}(x)-(2s+1)^2\sin x\,I_{s+\frac{1}{4}}(x)\}=\\ =\frac{2}{\pi}\Big|\frac{2}{x}\int_0^x I_0(x-\alpha)\sin\alpha\,.\sqrt{\alpha}\,d\alpha,\end{aligned}\right\} \tag{398}$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s\,I_{s+\frac{3}{4}}(x)\{(2s+2)^2\sin x\,I_{s+\frac{5}{4}}(x)+(2s+1)^2\cos x\,I_{s+\frac{1}{4}}(x)\}=\\ =\frac{2}{\pi}\Big|\frac{2}{x}\int_0^x I_0(x-\alpha)\cos\alpha\,.\sqrt{\alpha}\,d\alpha,\end{aligned}\right\} \tag{399}$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s\,I_{s+\frac{3}{4}}(x)\{(2s+1)\cos x\,I_{s+\frac{1}{4}}(x)-(2s+2)\sin x\,I_{s+\frac{5}{4}}(x)\}=\\ =\frac{2}{\pi}\Big|\frac{2}{x}\int_0^x I_0(x-\alpha)\cos 3\alpha\,.\sqrt{\alpha}\,d\alpha.\end{aligned}\right\} \tag{400}$$

Si nous multiplions les deux membres de (383) par $\sin 2x$ resp. $\cos 2x$ et ceux de (384′) par $\cos 2x$ resp. $\sin 2x$, alors nous trouvons après la

soustraction resp. l'addition des membres correspondants, et en faisant de même à l'égard de (385) et (386), ainsi de (387) et (388):

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)\,I_{\mu+s+\frac{1}{2}}(x)\{(2s+1)\sin 2x\,I_{\mu+s}(x)-(2s+3)\cos 2x\,I_{\mu+s+1}(x)\}=\\ =\frac{2}{\sqrt{\pi x}}\int_0^x I_{\mu-\frac{1}{2}}(\alpha)\,I_{\mu-\frac{1}{2}}(\alpha)\sin 2\alpha\,.\sqrt{\alpha\,(x-\alpha)}\,d\alpha,\end{aligned}\right\}\quad(401)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)\,I_{\mu+s+\frac{1}{2}}(x)\{(2s+1)\cos 2x\,I_{\mu+s}(x)+(2s+3)\sin 2x\,I_{\mu+s+1}(x)\}=\\ =\frac{2}{\sqrt{\pi x}}\int_0^x I_{\mu-\frac{1}{2}}(\alpha)\,I_{\mu-\frac{1}{2}}(\alpha)\cos 2\alpha\,.\sqrt{\alpha\,(x-\alpha)}\,d\alpha,\end{aligned}\right\}\quad(402)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)\,I_{s+\frac{1}{2}}(x)\{(2s+1)\sin 2x\,I_{s+\frac{1}{2}}(x)-(2s+3)\cos 2x\,I_{s+\frac{3}{2}}(x)\}=\\ =\frac{2}{\pi}\sqrt{\frac{2}{\pi x}}\int_0^x I_0(\alpha)\cos\alpha\sin 2\alpha\,.\sqrt{x-\alpha}\,d\alpha\end{aligned}\right\}\quad(403)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)\,I_{s+\frac{3}{2}}(x)\{(2s+1)\cos 2x\,I_{s+\frac{1}{2}}(x)+(2s+3)\sin 2x\,I_{s+\frac{3}{2}}(x)\}=\\ =\frac{2}{\pi}\sqrt{\frac{2}{\pi x}}\int_0^x I_0(\alpha)\cos\alpha\cos 2\alpha\,.\sqrt{x-\alpha}\,d\alpha,\end{aligned}\right\}\quad(404)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)\,I_{s+\frac{1}{2}}(x)\{(2s+1)\sin x\,I_{s+\frac{1}{2}}(x)-(2s+3)\cos x\,I_{s+\frac{3}{2}}(x)\}=\\ =\frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(\alpha)\sin\alpha\sin 2\alpha\,.\sqrt{x-\alpha}\,d\alpha,\end{aligned}\right\}\quad(405)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty}(-1)^s(s+1)\,I_{s+\frac{3}{2}}(x)\{(2s+1)\cos x\,I_{s+\frac{1}{2}}(x)+(2s+3)\sin x\,I_{s+\frac{3}{2}}(x)\}=\\ =\frac{2}{\pi}\sqrt{\frac{2}{x}}\int_0^x I_0(\alpha)\sin\alpha\cos 2\alpha\,.\sqrt{x-\alpha}\,d\alpha.\end{aligned}\right\}\quad(406)$$

§ 15. De la série connue [1]):

$$\sum_{n=0}^{\infty} \frac{\nu+\varrho+2n}{\nu+\varrho+n}\binom{\nu+\varrho+n}{n} I_{\nu+n}(x)\, I_{\varrho+n}(x) = \frac{\left(\frac{x}{2}\right)^{\nu+\varrho}}{\Gamma(\nu+1)\,\Gamma(\varrho+1)}$$

nous déduisons pour $\varrho = \nu - \frac{1}{2}$, en profitant de $\Gamma(\nu+1)\,\Gamma(\nu+\frac{1}{2}) = \frac{\Gamma(2\nu+1)\sqrt{\pi}}{2^{2\nu}}$, la particulière:

$$\sum_{n=0}^{\infty} \frac{2\nu+2n-\frac{1}{2}}{2\nu+n-\frac{1}{2}}\binom{2\nu+n-\frac{1}{2}}{n} I_{\nu+n}(x)\, I_{\nu+n-\frac{1}{2}}(x) = \frac{x^{2\nu-\frac{1}{2}}}{\Gamma(2\nu+1)\sqrt{\pi}}, \qquad \text{(V)}$$

à cause de laquelle nous remplaçons dans (263) ν par $\nu+n$ et multiplions les deux membres par $\frac{2\nu+2n-\frac{1}{2}}{2\nu+n-\frac{1}{2}}\binom{2\nu+n-\frac{1}{2}}{n}$; après la sommation sur n de 0 à ∞ et le changement des membres nous trouvons alors:

$$\sum_{n=0}^{\infty} \frac{2\nu+2n-\frac{1}{2}}{2\nu+n-\frac{1}{2}}\binom{2\nu+n-\frac{1}{2}}{n} I_{\nu+\varrho+n+\frac{1}{2}}(x)\, I_{\nu+\varrho+n}(x) =$$

$$= \frac{2\varrho+1}{\Gamma(2\nu+1)\sqrt{2\pi x}} \int_0^x I_{2\varrho+1}(2\alpha)\,(x-\alpha)^{2\nu}\,\frac{d\alpha}{\alpha}, \quad \begin{cases} R(\nu) > -\frac{1}{2} \\ R(\varrho) > -\frac{1}{2} \end{cases}$$

ou en posant $\nu = \mu - \varrho - \frac{1}{2}$:

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} \frac{2\mu-2\varrho+2n-\frac{3}{2}}{2\mu-2\varrho+n-\frac{3}{2}}\binom{2\mu-2\varrho+n-\frac{3}{2}}{n} I_{\mu+n}(x)\, I_{\mu+n-\frac{1}{2}}(x) = \\ &= \frac{2\varrho+1}{\Gamma(2\mu-2\varrho)\sqrt{2\pi x}} \int_0^x I_{2\varrho+1}(2\alpha)\,(x-\alpha)^{2\mu-2\varrho-1}\,\frac{d\alpha}{\alpha}, \quad R(\mu) > R(\varrho) > -\tfrac{1}{2}. \end{aligned}\right\} \quad (407)$$

La substitution $\mu = \varrho + \frac{3}{4}$ donne la formule particulière:

$$\sum_{n=0}^{\infty} I_{\varrho+n+\frac{3}{4}}(x)\, I_{\varrho+n+\frac{1}{4}}(x) = \frac{2\varrho+1}{\pi}\sqrt{\frac{2}{x}} \int_0^x I_{2\varrho+1}(2\alpha)\sqrt{x-\alpha}\,\frac{d\alpha}{\alpha}, \quad R(\varrho) > -\tfrac{1}{2}. \quad (408)$$

En substituant $\varrho = -\frac{1}{4}$ dans (407) et (408) on trouve:

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} \frac{2\mu+2n-1}{2\mu+n-1}\binom{2\mu+n-1}{n} I_{\mu+n}(x)\, I_{\mu+n-\frac{1}{2}}(x) = \\ &= \frac{1}{2\pi\,\Gamma(2\mu+\frac{1}{2})\sqrt{2x}} \int_0^x (x-\alpha)^{2\mu-\frac{1}{2}} \sin 2\alpha\,\frac{d\alpha}{\alpha\sqrt{\alpha}}, \quad R(\mu) > -\tfrac{1}{4} \end{aligned}\right\} \quad (409)$$

---

[1]) NIELSEN, l. c., p. 295, (6).

$$\left.\begin{aligned}\sum_{n=0}^{\infty} I_{n+\frac{1}{2}}(x)\, I_n(x) &= \frac{1}{\pi\sqrt{2\pi x}} \int_0^x \sin 2\alpha \,.\sqrt{\frac{x-\alpha}{\alpha}}\,\frac{d\alpha}{\alpha} = \\ &= \sqrt{\frac{2x}{\pi}}\,\{\cos x\, I_0(x) + \sin x\, I_1(x)\} - \frac{1}{\sqrt{2\pi x}} \sin x\, I_0(x) \text{ selon (107).}\end{aligned}\right\} \quad (410)$$

Ayant égard à (343) on trouve donc:

$$\int_0^x I_0(\alpha)\cos\alpha\, d\alpha = x\,\{\cos x\, I_0(x) + \sin x\, I_1(x)\} - \tfrac{1}{2}\sin x\, I_0(x), \quad (411)$$

$$\int_0^x I_1(\alpha)\sin\alpha\, d\alpha = x\,\{\cos x\, I_0(x) + \sin x\, I_1(x)\} - \tfrac{3}{2}\sin x\, I_0(x). \quad (412)$$

La substitution $\mu = \frac{1}{4}$ dans (409) donne:

$$\sum_{n=0}^{\infty} \frac{4n-1}{2n-1}\binom{n-\frac{1}{2}}{n} I_{n+\frac{1}{2}}(x)\, I_{n-\frac{1}{2}}(x) = \frac{1}{2\pi\sqrt{2\pi x}} \int_0^x \sin 2\alpha\, \frac{d\alpha}{\alpha\sqrt{\alpha}}. \quad (413)$$

Si nous remplaçons, à cause de I, dans (408) ϱ par $\varrho + m$ et multiplions les deux membres par $(-1)^m$, alors nous trouvons après la sommation sur m de 0 à ∞, en appliquant le théorème de CAUCHY (g):

$$\sum_{s=0}^{\infty} (-1)^s I_{\varrho+2s+\frac{1}{2}}(x)\, I_{\varrho+2s+\frac{1}{2}}(x) = \frac{1}{\pi}\sqrt{\frac{2}{x}} \int_0^x I_{2\varrho}(2\alpha)\,.\sqrt{x-\alpha}\, d\alpha, \; R(\varrho) > -\tfrac{1}{2} \quad (414)$$

de laquelle suivent pour $\varrho = -\frac{1}{4}$ et $\varrho = \frac{1}{4}$:

$$\left.\begin{aligned}\sum_{s=0}^{\infty} (-1)^s I_{2s+\frac{1}{2}}(x)\, I_{2s}(x) &= \frac{1}{\pi}\sqrt{\frac{2}{\pi x}} \int_0^x \cos 2\alpha\,.\sqrt{\frac{x-\alpha}{\alpha}}\, d\alpha = \\ &= \sqrt{\frac{x}{2\pi}}\,\{\cos x\, I_0(x) + \sin x\, I_1(x)\} \text{ en vertu de (230),}\end{aligned}\right\} \quad (415)$$

$$\left.\begin{aligned}\sum_{s=0}^{\infty} (-1)^s I_{2s+1}(x)\, I_{2s+\frac{1}{2}}(x) &= \frac{1}{\pi}\sqrt{\frac{2}{\pi x}} \int_0^x \sin 2\alpha\,.\sqrt{\frac{x-\alpha}{\alpha}}\, d\alpha = \\ &= \sqrt{\frac{x}{2\pi}}\,\{\sin x\, I_0(x) - \cos x\, I_1(x)\} \text{ en vertu de (229).}\end{aligned}\right\} \quad (416)$$

Biochemistry. — *Motory phenomena in coacervate drops in a diffusion field and in the electric field.* By H. G. BUNGENBERG DE JONG and E. G. HOSKAM. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of September 27, 1941.)

1. *Introduction.*

In a direct current field drops of complex coacervates show a great many phenomena:

a. shifting with regard to the medium (electrophoresis),

b. transformation (flattening to discs perpendicular to the direction of the field, BÜCHNER effect),

c. vacuolization and motory phenomena in the internal of the coacervate drops,

d. displacement of larger vacuoles and enclosures (e.g. oildrops, carbon particles, coexisting coacervate drops) to one side of the coacervate drop. When the surface is reached the vacuoles break, the enclosures only bulge out, but when the current is discontinued they return to the inside of the drop,

e. formation of new, small coacervate drops in the medium on the other side of the drop as in *d*, at some distance from the surface.

Of the above effects those mentioned under *a*, *c*, *d* and *e* are typically polar effects, they are reversed when the polarity of the electric field is changed, moreover, in a positive drop they are the reflected images of those in a negative one.

In a previous publication [1]) we have given a preliminary explanation of the phenomena mentioned in *c*, *d* and *e*, which together we named desintegration phenomena. In this explanation we started from electro-kinetic phenomena (electrophoresis and electroendosmosis). Further study of the phenomena mentioned under *c* reveals that the above starting point presents difficulties when we try to account for the desintegration. It has appeared that analogous phenomena may also occur in a diffusion field, so that we may suspect a connection between the two. This suggests a new starting point for the explanation of desintegration, in which an essential part is ascribed to the changes in the interfacial tension: coacervatedrop-medium.

2. *Motory phenomena in a diffusion field.*

The phenomena which will be described in this section were observed at room temperature in gelatine containing coacervates in which the gelation had been neutralized by the addition of suitable non-electrolytes.

We started from a stock solution consisting of 6 g. gum arabic + 5 g. gelatine + 190 cc dist. H_2O. 10 cc of this stock solution was added to a mixture consisting of 2 g. resorcine + 3 cc HCl 0.1 N + 10 cc dist. H_2O. The result is a coacervation which continues to exist after cooling to room temperature.

A little of the coacervated system was placed on an object glass, which was provided with a thin film of a dried up amylum-soluble solution. This prevents the coacervate drops from spreading over the glass surface. When a small granule of NaCl is carefully put in the fluid, currents are formed in the near coacervate drops. On the side where the coacervate drop comes into contact with the NaCl which diffuses towards it, the coacervate becomes more dense (i.e. richer in colloids), considerable vacuolization setting in.

Now the surface of the coacervate drop is seen to move in the direction where the coacervate becomes more dense. In consequence of it the entire contents of the coacervate begin to move, as is seen from the motion of the vacuoles which are carried with it (see fig. 1A). Moreover the coacervate drop creeps away in the direction opposite to that from where the NaCl diffuses.

---

[1]) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Biochem. Z. **221**, 403 (1930); Kolloid-Beihefte **43**, 143 (1935).

The reverse picture too may be obtained, namely that all the motory phenomena are reversed. This happens when the coacervate drop comes into one-sided contact with a "swelling" medium.

In order to make the current distinctly visible, care should be taken to provide the coacervate drop equally with small vacuoles. This is done by making the coacervate more dense just before starting the experiment.

The following method proved effective: 10 cc of the above basis solution is added to a mixture of 1 g. resorcine + 2.5 g. ureum + 2.5 cc HCl 0.1 N + 9 cc dist. water. A complex coacervation is formed which continues to exist on cooling to room temperature. A little of the coacervated system is placed on an object glass prepared with amylum-solubile, but the fine, perfectly hyalin coacervate drops are not yet fit for the experiment. We now bring about an equal, very fine vacuolization, placing a drop of distilled water on the preparation. When then we introduce a granule of NaCl in the preparation the reverse happens as in fig. 1A (see fig. 1B). The disappearance of the fine vacuolization

Fig. 1. Current phenomena in a diffusion field of a condensing (left) resp. opening substance (right). The direction into which this substance diffuses is indicated by an arrow by the side of the drops. The direction in which the coacervate drops creep over the substratum is indicated by an arrow under the drops.

(the coacervate becomes hyaline in that spot) marks the place where the "swelling medium" touches the coacervate, while the direction of the current is indicated by the motion of the remaining vacuoles. The coacervate drop moreover creeps in the direction from where the opening substance diffuses.

The motory phenomena described above may be explained when we assume that condensation of the coacervate is combined with increase of the interfacial tension: coacervate/medium, reversely that "opening" lowers that tension. For then the phase boundary must move from places with less, to places with higher interfacial tension. The supposition made which would seem plausible enough, could not be tested by direct method, as the precise measuring of the very slight interfacial tension characteristic of very liquid coacervates, yet presents great difficulties.

An indirect indication in favour of the supposition made is the transformation of the coacervate drop in the diffusion field, which change is not pictured in the schemes of fig. 1A and 1B. On vertical view the coacervate drops are originally perfectly circular, but in the case of fig. 1A they assume the shape of fig. 2A; in the case of fig. 1B that of fig. 2B.

Fig. 2. Transformation of the coacervate drops during the one-sided activity of a condensing (left) resp. opening (right) substance. The direction into which this substance diffuses is indicated by an arrow.

That is to say, on the side where the coacervate becomes relatively most dense, the curve radius of the drop is greater than on the opposite side. Such a shape must also be assumed by the drop when the interfacial tension is different on the right and on the left, namely exactly as we supposed in the above (the greatest curve radius on the side of the greatest interfacial tension).

An answer to the question why NaCl has an opposite effect in the two cases discussed above may here be suggested in a few words. In the case discussed in fig. 1B we see the normal behaviour of the complex coacervates: in sufficient concentration neutral salts cause neutralization and in the previous concentration section the percentage of the dispersing medium in the coacervate increases ("swelling of the coacervate"). From a special investigation it has appeared that in the presence of 10 % ureum + 4 % resorcine the complex coacervation at room temperature comes very near that at 40° in the absence of the two non-electrolytes.

In the case of fig. 1A we only used resorcine as a gelation neutralizing agent. Certain secundary effects of the resorcine, especially on the gelatine component have not been neutralized here by ureum and are the cause that the neutralizing resp. swelling effect of the neutral salts on complex coacervates are dominated by the condensing effect of salts, as it is also expressed in the formation of coacervates from resorcine-containing positive gelatine sols with neutral salts.

3. *Current and vacuolization phenomena of the complex coacervate in the electric field.*

The desintegration phenomena (see introduction) have formerly been studied in coacervate drops floating free in the medium. In these methods the observation of what takes place first in the electric field is very difficult, as the coacervate drops continually disappear from the field of vision. Consequently we were pretty well acquainted with the facts mentioned in *d* and *e* in the introduction, whereas an important detail of point *c* had escaped our attention.

Undisturbed observation becomes possible by studying coacervate drops on a starched object glass on a heated object table. The auxiliary apparatus pictured in fig. 3, which is screwed on to the objective, serves to protect the objective and to fix the electrodes.

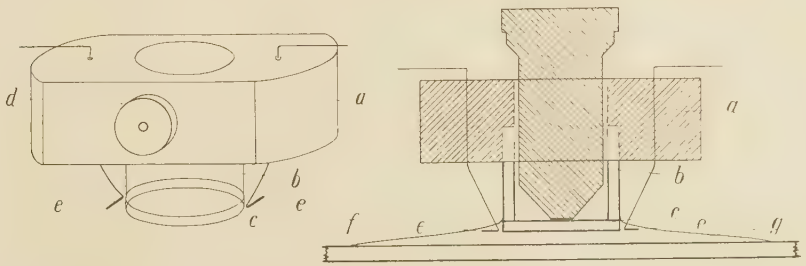


Fig. 3. Auxiliary apparatus to be fixed to the objective. a) ebonite block; b) glass tube to which has been cemented; c) a glass plate; d) screw; e) two platinum electrodes; f) object glass; g) liquid containing the coacervate drops.

The complex coacervate of gelatine and gum arabic were studied at 40°. The gelatine (F 00 extra of the "Lijm- en Gelatinefabriek „Delft" " at Delft) was purified by LOEB's method. As gum arabic we used Na-arabinate [1]) 5 g. of both colloids was dissolved in 250 cc dist. water. Positive, uncharged and negative coacervates were obtained by mixing a cc gelatine + (10–a) cc arabinate with 5 cc of an acetate buffer of the following

1) Bioch. Z. **59**, 436 (1933); **262**, 161 (1933).

composition: to 30 cc Na-acetate 0.1 N + 25 cc acetic acid. 2 N dist. water was added until 100 cc. The buffer by itself at 40° had a pH value of 3.49. In the above mixtures the pH varies between 3.65 (only gelatine present) and 3.57 (only Na-arabinate present), so that the series of mixtures was approximately isohydric.

The following table gives the coacervate volumes and the electrophoretic velocity (in arbitrary units at 40°), as function of the mixing proportion [2]).

Mixing prop. in % arabinate.	Coacervate vol. in 0.1 cc	Electrophoretic velocity in arbitrary units and charge indicator sign.
20	0.2	
30	3.4	
40	6.8	+ 131
45	8.3	+ 50 reversal of charge at 48 % A.
50	9.1	− 17
55	9.2	− 174
60	8.6	
70	4.8	
80	0.1	

The peak of the coacervate volume curve is between 50 and 55 % A, while reverse of charge takes place at a slightly lower mixing proportion (48 %).

When by the method described and in weak fields (5 to 10 volts/c.M.) we investigate coacervated systems with 30 resp. 40 % arabinate (positive), resp. with 60 and 70 % arabinate (negative), then, as is seen from the motion of the vacuoles formed, a current is formed in the coacervate drops as shown in fig. 4. Here too, vacuoles are formed in the place where the peripheral motion turns inward.

Fig. 4. Current phenomena in a complex coacervate drop in an electric field.

Nevertheless the phenomena are more complicated than in the diffusion field of opening or condensing substances. For there is also formed an independent, second system of usually very small vacuoles on the opposite side of the drop. The second system is found exactly under the interface: coacervate/medium.

A short time after the field is applied we obtain pictures as in fig. 5. When coacervated systems are examined very near the point of reversal of charge, the current phenomena do not cease, but on the contrary, show a more complicated type: in the left half of the

Fig. 5. Vacuolization a short time after the application of the electric field.

[2]) In this place we wish to express our thanks to Dr. H. L. BOOY for taking the electrophoresis measurements.

drop there is the picture of the current as in positive drops, in the right half there is the picture of negative drops (see fig. 4 and 5).

Finally we observe that, as in the diffusion field, so in the electric field, the coacervate drops creep slowly over the substratum, namely the positive drops to the — pole, the negative ones to the + pole.

4. *Discussion.*

For the theory of the desintegration phenomena the following results are of importance:

a. The type of the current phenomena in the electric field in drops with a strongly positive and in drops with a strongly negative charge is the same as in a diffusion field of a condensing or opening substance.

b. In uncharged drops there is a more complicated type of current.

Formerly we thought that in explaining the desintegration phenomena we must start from the independent electrophoresis of the two biocolloids in the coacervate with opposite charge. The current phenomena we also observed we thought attributable to electroendosmosis of the coacervate along the bounding face coacervate/medium (considered as a wall with the same electrophoresis sign as the drop has with regard to the medium). Now that we have been able so much the better to observe the initial phenomena in the electric field, this interpretation must be considered incorrect.

In the first place it appears from the creeping phenomena that the surface of the coacervate drop itself shifts, and that in the same direction as the peripheral layers of the coacervate.

Secondly the fact that the current type in uncharged drops becomes more complicated is contradictory to the theory of electroendosmosis (the current should cease here).

On the other hand the results indicate that polarization at the interface coacervate/medium must be considered the chief cause of the current. It may be supposed that the attending pH changes cause condensation on the one side of the drop, and opening on the opposite side. The consequent changes of the interfacial tension will then cause the contents of the coacervate drop to flow in the same way as was indicated in fig. 1 for coacervate drops in a diffusion field of a condensing or opening substance.

The more complicated type of current in practically uncharged drops agrees very well with this interpretation. Again we start from the plausible supposition that there is an intimate connection between effective attraction and interfacial tension: on "condensation" the latter will increase, on opening however, it will decrease. As at the reversal of charge the effective attraction is maximal, and therefore pH decrease as well as pH increase will cause the effective attraction to fall, the interfacial tension of the uncharged coacervate drop will also be maximal and the pH changes mentioned can only decrease the interfacial tension. When polarization occurs in the electric field the pH rises on the one side and falls on the other side of the coacervate drop, but these pH changes both cause decrease of the interfacial tension in the uncharged coacervate drop. As the bounding face will begin to shift from places with a lower interfacial tension (the two "poles" of the drop, located in the direction of the electric field) to places with a higher interfacial tension (the equator of the drop, perpendicular to the electric field) the current type of fig. 4 and 5 must develop in the uncharged coacervate drop.

We realize that we can only roughly account for the current phenomena in the electric field. Many details, such as the second peripheral system of vacuoles and the changes of the current figure with time require further study, so that as yet we cannot attempt an explanation of these phenomena.

Leiden, Laboratory for Medical Chemistry.

Biochemistry. — *The influence of the temperature on the complexcoacervation of gelatine and gum arabic.* By H. G. BUNGENBERG DE JONG, E. G. HOSKAM and L. H. V. D. BRANDHOF–SCHAEGEN. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of September 27, 1941.)

1. *Introduction.*

In a mixture of gelatine and gum arabic sols acidification causes de-mixing ("coacervation") into a liquid layer rich in colloids (complexcoacervate) and a liquid poor in colloids ("equilibrium liquid").

A detailed investigation [1]) of this kind of coacervation has previously been made at constant temperature, which showed a.o. from the occurrence of optimal coacervation at different mixing proportions of the isohydric sols for each pH, that the opposition of charge (gelatine positive, gum arabic negative) is the most important factor in the process of complexcoacervation. We now put the question what may be the influence of the temperature on complexcoacervation. So on the one hand we investigated whether the optimal mixing proportion shifts with the temperature, on the other hand we were interested in any changes of the colloid percentage of the equilibrium liquid ("solubility of the coacervate") and of the percentage of water in the coacervate. Finally we examined the two latter points in the presence of a neutral salt (KCl).

2. *Methods.*

3% stock solutions: 30 g. fine ground gum arabic (gomme senegal petite boule blanche I) is dissolved by heating in 1 L dist. water and filtered. The stock is kept in the refrigerator. 15 g. gelatine (gelatine for bacteriologic purposes of the Lijm en Gelatinefabriek "Delft" at Delft) is strewn on 500 cc dist. water and left to swell for 30 min. Then it is dissolved in a water bath at 60°. The solution is divided into portions, which are preserved in the refrigerator. On use a portion is melted and placed in a burette with a hot water mantle.

For the preparation of the coacervated systems a number of wide tubes, to whose lower ends narrower tubes are melted — divided into 0.1 cc — is placed in the thermostat and filled with a cc 3% gum arabic solution + (20-a) cc 3% gelatine solution. After the mixtures have reached the desired temperature 5 cc of a 4% acetic acid solution is added to each tube and after the tubes have been closed with a rubber stop the contents are well mixed. After 1 hour in the thermostat the mixtures are shaken again and then left till the next morning. Now the volume of the coacervate layers is noted and samples (ca. 10 cc) of the equilibrium liquid are taken in order to determine the dryweight. They are placed in cylindrical nickle dishes (diameter 70 mm, height 18 mm) weighed and then dried, one hour on boiling water bath, then one hour in the drying oven at 105°. In the later experiments (see §§ 6 and 7) samples were also taken of the coacervate layers, in order to determine the percentage of water of the coacervate.

3. *Optimal mixing proportion and temperature.*

In Table I are found the results of two, identical series at 40°. The only difference is that the second series was begun a few days after the first. They make the impression of being reproducible.

The dryweight determinations were begun in duplicate only in a few mixing proportions. In the experimental series of Table II, in which we started from new stock solutions of

[1]) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Kolloid Beihefte **43**, 143 (1935); **43**, 213 (1936).

gelatine and gum arabic, the dryweight determinations were always done in duplicate. From the data of Tables I and II we can answer the question whether the temperature affects the mixing proportion of optimal coacervation.

The mixing proportion is indicated on the one hand by the maximum of the coacervation-volume curve, on the other hand by the minimal dryweight within each mixing series.

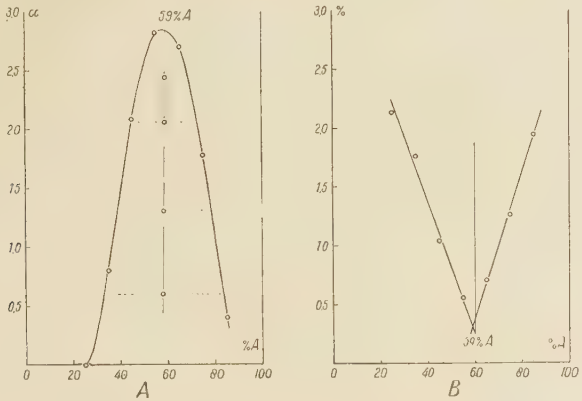


Fig. 1. Coacervate volume (A) and dryweight of the equilibrium liquid (B) as functions of the mixing proportions (in % arabinate), according to the data of the experimental series at 50° of Table II. As maximum of the curve in fig. 1 A we assume the point of dissection of the bisecting line and the curve. In fig. 1 B straight lines are drawn through the experimental points, and the mixing proportion near the point of dissection is determined. This mixing proportion is assumed to be the mixing proportion at which the dryweight of the equilibrium liquid reaches a minimum.

TABLE I.

	cc A	cc G	% A	Coac. vol. in cc	% dryweight equilibrium liquid
40° Series I	5	15	25	0.02	2.11 / 2.12 } 2.11
	7	13	35	0.69	1.76
	9	11	45	2.10	0.99
	11	9	55	2.85	0.52 / 0.48 } 0.50
	13	7	65	2.74	0.62 / 0.61 } 0 62
	15	5	75	1.80	1.20
	17	3	85	0.33	
40° Series II	5	15	25	0.01	
	7	13	35	0.71	1.79 / 1.82 } 1.81
	9	11	45	2.13	1.00
	11	9	55	2.85	0.46
	13	7	65	2.83	0.59
	15	5	75	1.80	1.25
	17	3	85	0.38	1.90

The former maximum cannot easily be determined directly, as the maximum sections of the curves are rather level. We are assisted by the construction of a bisecting line (see fig. 1 A), in which we use the steep curve branches. It is found that in constructing the coacervate volume curves from the data of Table I and II the bisecting lines intersect the maximum sections at 58—59 % gum arabic, though the temperatures are different (31°, 40° and 50°).

With greater certainty the optimal mixing proportion is found from the dryweights of the equilibrium liquid (fig. 1B), because the two branches of the curves over a longer section are practically straight lines. Here again, for all four curves the point of dissection of those straight lines is near 59 % gum arabic. *So we conclude that the optimal mixing proportion is independent of the temperature.*

TABLE II.

	cc A	cc G	% A	Coac. vol. in cc	% dryweight equilibrium liquid
31°	0	20	0		2.05 / 2.10 } 2.08
	5	15	25		2.09 / 2.09 } 2.09
	7	13	35	0.50	1.61 / 1.59 } 1.60
	9	11	45	2.08	0.92 / 0.91 } 0.92
	11	9	55	2.64	0.45 / 0.47 } 0.46
	13	7	65	2.66	0.59 / 0.58 } 0.59
	15	5	75	1.75	1.18 / 1.22 } 1.20
	17	3	85	0.48	1.84 / 1.83 } 1.84
	20	0	100		2.05 / 2.07 } 2.06
50°	5	15	25	0.01	2.14 / 2.11 } 2.13
	7	13	35	0.80	1.78 / 1.74 } 1.76
	9	11	45	2.09	1.04 / 1.03 } 1.04
	11	9	55	2.82	0.56 / 0.55 } 0.56
	13	7	65	2.70	0.70 / 0.71 } 0.71
	15	5	75	1.78	1.28 / 1.26 } 1.27
	17	3	85	0.40	1.94 / 1.96 } 1.95

4. *Coacervate volume and temperature.*

From the comparison of the results of Table II it would follow that with the same mixing proportion the coacervate volume is a little greater at 50° than at 31°. Although these series of experiments were begun with the same stock solutions, they are not strictly

comparable, because they were not begun simultaneously. Therefore from now on we mounted 3 thermostats and experimental series were begun simultaneously with the same stock solutions, at two or three temperatures. We restrict ourselves to the section of mixing proportions in which the coarcervate volume is great. From table III it follows that the coacervate volume at 50° is a little smaller than at 30°, whereas there is practically no difference averagely between the volumes at 30° and at 40°.

TABLE III.

cc A	cc G	% A	Coacervate volume in cc			% dryweight in equil. liquid		
			30°	40°	50°	30°	40°	50°
9	11	45	2.21	2.23	2.17	0.98	0.99	1.10
9	11	45	2.17	2.21	2.16	1.03	1.08	1.08
11	9	55	2.90	2.94	2.89	0.54	0.53	0.58
11	9	55	2.96	2.90	2.82	0.55	0.52	0.62
13	7	65	2.82	2.86	2.72	0.65	0.65	0.73
13	7	65	2.85	2.82	2.72	0.69	0.64	0.69

In a second, similar series (Table IV) we also see hardly any difference between 31.25 ° and 40°.

TABLE IV.

cc A	cc G	% A	Coacervate volume in cc		% dryweight equil. liquid	
			31.25°	40°	31.25°	40°
11	9	55	2.89	2.91	0.49	0.53
11	9	55	2.87	2.91	—	0.53
12	8	60	2.94	2.95	0 47	0.53
12	8	60	2.93	2.92	0.49	0.53
12	8	60	2.92	2.95	0.47	0.50
12	8	60	2.96	2.96	0.47	0.50
13	7	65	2.71	2.79	0.64	0.65
13	7	65	2.74	2.75	0.64	0.66

Also in the subsequent experiments we see (tables V and VI, further in this paper) that the effect sought is of the same order as the experimental errors, so that we may say that at least between 30° to 40° the coacervate volume is very little affected by the temperature.

In the presence, however, of KCl (5.10 and 20 m Aeq. p. L), Table VI clearly shows that the coacervate volume decreases at a rise in temperature. But this is likewise always the case in the temperature section of 40°—50°, as is seen from Tables III, V and VI.

The vague results in the section of 30°—40° are probably connected with the fact that at 30° we are very near the gelation temperature. The following experiment makes this clear. After being left for a night a clear coacervate was cooled very slowly (the flame under the thermostat being turned out). At 28.5° the coacervate layer begins to get cloudy. At further cooling the result is a turbid gelatinized complexcoacervate layer. From 16° onward the thermostat was slowly heated and then the cloudiness only disappeared entirely at 31.6°. So from the point of view of onsetting internal changes in the coaservate we should avoid temperatures between 28.5° and 31.6°, when drawing conclusions concerning the effect of the temperature on ideal liquid coacervates and possibly we are too near this temperature section also at 33°.

Hence we arrive at the conclusion that the *coacervate volume in a temperature section, sufficiently removed from the gelation temperature slightly decreases at an increase of temperature.*

Microscopic examination of the gelatinized complexcoacervates shows that the cloudiness is caused by the presence of vacuoles.

On rapid cooling many small vacuoles are formed, whereas in slow cooling the gelatinized coacervate contains a smaller number of larger vacuoles. Apparently the vacuoles are formed during the semi-liquid condition of the coacervate, so that the confluence of the vacuoles is still possible.

5. *Solubility of the coacervate and temperature.*

When we take the averages of the dryweights of the equilibrium liquid of the two series in Table I, comparing them with those in Table II, we note an evident increase in the solubility of the coacervate with the temperature.

Survey belonging to Tables I and II.

% A	Dryweight equilibrium liquid in %.		
	31°	40°	50°
35	1.60	1.79	1.76
45	0.92	1.00	1.04
55	0.46	0.48	0.56
65	0.59	0.61	0.71
75	1.20	1.23	1.27
85	1.84	1.90	1.95

The data of Table III do, indeed, show the slight increase of the dryweights between 40° and 50°, but the dryweights of 30° to 40° do not allow of any conclusion. Table IV shows an evident increase in the solubility with the temperature rise, with regard to section 31.3°—40°. The same is seen in the results of Tables V and VI, further in this paper, in which we compare the complex coacervation at 33.2°—40° and at 50°. The deviating result of Table III we are inclined to attribute to the fact that the lowest temperature used here (30°) is too near the gelatinization temperature.

So we arrive at the conclusion that *the solubility of the coacervate in a temperature section sufficiently removed from the gelatinization temperature slightly increases with a rise in the temperature.*

6. *Percentage of water of the coacervate and temperature.*

In the following series of experiments the mixing proportions of 55 and 60% gum arabic were begun in quadruplicate, while the dryweights not only of the equilibrium liquids, but also of the coacervate layers were determined. The samples of the coacervates were taken with small pipettes, which had been closed by melting the lower ends. The fine points were scratched with a glass file. First part of the equilibrium liquid was sucked off for the analysis of this liquid, after which the pipettes were placed in the sedimentation tubes for one hour, then the point was broken off by slight pressure. The coacervate then flowed in very slowly under hydrostatic pressure, or in a few cases was helped to flow by sucking with slight lower pressure. The latter should be done with great care, as otherwise there is a danger of the equilibrium liquid also entering the pipette in consequence of vortex formation.

The samples of the coacervate layer were pipetted directly into the nickle drying boxes and weighed at once. After that they were distributed equally over the bottom with hot distilled water, and then dried.

TABLE V.

cc A	cc G	% A	Coacervate vol. in cc			% dryweight equil. liquid			% dryweight coacervate		
			33.2°	40°	50°	33.2°	40°	50°	33.2°	40°	50°
11	9	55	2.79	2.79	2.70	0.49	0.53	0.58	—	—	—
11	9	55	2.82	2.82	2.73	0.49	0.52	0.63	12.88	13.02	13.00
11	9	55	2.80	2.79	2.75	0.49	0.51	0.64	13.19	13.02	13.12
11	9	55	2.82	2.83	2.73	0.50	0.53	0.57	12.83	13.16	13.05
12	8	60	2 87	2.90	2.78	0.53	0.55	0.66	12.83	13.00	12.96
12	8	60	2.87	2.81	2 75	0.53	0.55	0.68	12.84	13.06	—
12	8	60	2.89	2.87	2.79	0.52	0.55	0.67	12.88	12.95	13.00
12	8	60	2.85	2.89	2.77	0.54	0.57	0.68	12.92	12.97	13.05

The results of this experimental series are found in Table V. Below we give a survey stating the averages of the quadruplicate determinations.

Survey belonging to Table V.

% A	Coacervate volume in cc			% Dryweight equil. liquid			% Dryweight coacervate		
	33.2°	40°	50°	33.2°	40°	50°	33.2°	40°	50°
55	2.81	2.81	2.73	0.49	0.52	0.61	12.86[1])	13.07	13.06
60	2.87	2.86	2.77	0.53	0.55	0.67	12.87	12.99	13.00

The influence of the temperature on the coacervate volume and on the solubility of the coacervate has been discussed in previous sections, so that we restrict ourselves to the dryweight of the coacervate. From the list is seen a slight increase of the dryweight at a temperature increase in the section of 33.2°—50°. The same is seen in Table VI in the absence of KCl.

But there the value is at 40° (13.20 %), between those at 33.2° (12.98 %) and at 50° (13.39 %), whereas in Table V the value of 40° is equal to that of 50°. Also in the presence of KCl (5.10 and resp. 20 m aeq. p. L) we always see (in Table VI) that the dryweight of the coacervate is greater at 50° than at 33.2°, while at 40° the value is either a little more or a little less than at 33.2°. We think we must attribute the varying results with regard to the emperaure secion of 33.2°—40° to experimental errors. As is also seen from the tables, the separate values within each quadruplicate sometimes fluctuate over a greater amount than the difference of the averages of 33.2° and 40°.

So we arrive at the conclusion that *the percentage of water of the coacervate depends but little on the temperature, decreasing very slightly at an increase of temperature* (the dryweight slightly increasing).

7. *The effect of the temperature on the complexcoacervation in the presence of a neutral salt (KCl).*

Neutral salts have a neutralizing effect on complexcoacervation. They increase the solubility of the coacervate and make it richer in water. We investigated whether the presence of KCl affects the dependence on temperature of the coacervate volume, the solubility of the coacervate and its percentage of water, discussed in the foregoing pages. This experiment was done in one mixing proportion (60 % gum arabic), of the two colloids. The mixtures were made as follows: 25 cc solution of gum arabic and gelatine

[1]) The dubious value of 13.19 was not used in determining the average.

+ 2 cc a-normal KCl + 5 cc 4% acetic acid. The KCl solution was selected so that the final concentration in the mixtures was 0.5, 10.20 and 40 m aeq. p. L.

At 40 m. aeq. p. L. the coacervation was already neutralized. The results are found in Table VI. The averages are given in the following survey.

This survey shows the typical neutralizing effect of a neutral salt on complex-coacervation: the solubility of the coacervate increases (% dryweight of the equilibrium liquid increases) and the percentage of water of the coacervate increases (% dryweight of the coacervate decreases).

The presence of KCl, however, does not cause any change in the influence of the temperature on complexcoacervation. When the temperature is raised the coacervate

TABLE VI.

KCl concentration m. aeq. p. L.	Coacervate vol. in cc			% Dryweight equil. liquid			% Dryweight coacervate		
	33.2°	40°	50°	33.2°	40°	50°	33.2°	40°	50°
0	3.67	3.67	3.54	0.48	0.50	0.53	12.98	13.15	13.39
		3.70			0.50			13.24	
5	3.75	3.75	3.68	0.65	0.71	0.74	12.01	12.02	12.22
	3.69	3.76	3.60	0.65	0.68	0.72		12.02	
	3.75	3.77	3.60	0.65	0.69	0.73	11.98	12.03	12.13
10	3.75	3.68	3.56	0.90	0.98	0.99	10.90	10.89	11.10
	3.69	3.68	3.59	0.91	0.98	0.99	11.05	10.93	11.15
	3.72	3.68	3.54	0.91	0.94	1.01	10.98	10.89	11.22
		3.68	3.52		0.95	0.99			11.14
20	2.79	2.58	2.40	1.54	1.63	1.72	8.90	8.77	8.99
	2.82	2.57	2.45	1.53	1.62	1.73	8.85	8.95	9.12
	2.83	2.58	2.40	1.53	1.65	1.69	8.99	8.80	9.15
	2.87		2.40	1.51		1.73	9.06		9.04

volume slightly decreases (see § 4), the solubility of the coacervate slightly increases (see § 5) and the percentage of water of the coacervate slightly decreases (see § 6).

Survey belonging to Table VI.

KCl m. aeq. p. L.	Coacervate vol. in cc			% Dryweight equil. liquid			% Dryweight coacervate		
	33.2°	40°	50°	33.2°	40°	50°	33.2°	40°	50°
0	3.67	3.68	3.54	0.48	0.50	0.53	12.98	13.20	13.39
5	3.72	3.76	3.63	0.65	0.69	0.73	12.00	12.02	12.17
10	3 72	3.68	3.55	0.91	0.96	0.99	10.98	10.90	11.15
20	2.83	2.58	2.41	1.53	1.63	1.72	8.95	8.84	9.05

8. *Summary and discussion.*

In the foregoing pages we arrived at the conclusion that when the temperature is raised:

a. the optimal mixing proportion remains constant,

b. the coacervate volume decreases slightly,

c. the solubility of the coacervate slightly increases,

d. the percentage of water in the coacervate slightly decreases.

As for these results, *a.* need not surprise us, as the optimal mixing proportion is determined by the (effective) equivalent weight of the two colloidions, and this is not likely to be greatly dependent on the temperature.

Concerning *b.* we remark that the influence of the temperature on the coacervate volume will be the result of the influence of the temperature on the solubility (*c*) and on the percentage of water of the coacervate (*d*). The result mentioned under *b.* follows from the increasing solubility and the decreasing percentage of water of the coacervate.

The most remarkable result is the decrease of the percentage of water of the coacervate d., while at the same time its solubility increases (d).

We have before arrived at the conclusion that the internal condition of a complex coacervate is determined by two opposing factors: 1. an electrostatic attraction of colloidions of opposite charge, 2. a repulsion factor connected with the hydratation of the two colloidions. The dependence on the temperature of the percentage of water of the coacervate we are therefore inclined to attribute to a change in the repulsion factor mentioned under 2. For the hydratation will decrease when the temperature is raised and with it the repulsion factor will grow less.

Besides there is the general influence of the temperature, on the ground of which we may expect the two coexisting liquids (coacervate and equilibrium liquid) to mix better when the temperature is increased, i.e. the equilibrium liquid will contain more colloids, the coacervate more water.

The temperature effect mentioned first is entirely absent in the equilibrium liquid, so that the solubility of the coacervate wll increase at a rise in the temperature (*c*). On the other hand this effect does play a part in the coacervate, and in view of the decreasing percentage of water on a rise of temperature (*d*) we must assume that here it decreases the percentage of water to a greater extent, than the water percentage would increase in consequence of the general temperature effect, mentioned last.

The results are also of interest on account of another problem. We saw in § 4 that on gelation vacuolization occurs in the semi-liquid coacervate. This means that the coacervate loses water. Now two things may be supposed: *a*) the vacuolization is independent of the gelation, but is the continuation of a temperature effect on the percentage of water of the coacervate, which exists already at much higher temperatures, *b*) the vacuolization is directly connected with gelation. The results show that *a*) is not correct. For in the temperature section between 50°—40° the coacervate does not become poorer, but richer in water on cooling. In favour of *b*) is also the fact that the forming of vacuoles is a process which may also occur isothermically at a suitable temperature, lower than the gelation temperature.

Leiden, Laboratory for Medical Chemistry.

Botany. — *Composition of Gas in the Intercellular Spaces of Potatoes.* By A. GORTER and W. NADORT. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of October 25, 1941.)

Introduction.

Studying the physiological phenomenons of potatoes, various conclusions have from time to time been drawn from the composition of the gases in the intercellular spaces, the internal atmosphere. Amongst others, THORNTON (8), in a recent publication on factors influencing the sprouting of potato-buds, makes use of the composition of the gas in the intercellular spaces to prove that sprouting is promoted by lack of oxygen.

As early as 1890 extensive investigations into the internal atmosphere were made by DEVAUX (3), but these have evidently been forgotten meanwhile, for MAGNESS (4) writes in 1920: "We have no critical studies upon the internal gases of plant tissues.....". He describes a method to determine the composition of the inner atmosphere and reports some analyses from which he finds a composition quite different from that found by DEVAUX. In this article we shall prove that MAGNESS' method does not give good results, and that to determine the composition of the gas by this method, as, e.g. DAVIS (2) and SMITH (7) have done, easily leads to wrong conclusions. We shall first discuss both methods briefly.

Method of DEVAUX.

During microscopical studies upon potatoes, this investigator pointed out that the very small intercellular spaces all communicate with each other, while, too, with a slight underpressure air can easily be sucked through the potato. From this he concluded that the tuber must be very porous, and that an equilibrium exists between the gas and the fluid in the intercellular spaces. This equilibrium will be restored at any time it is disturbed, even if this may take rather a long while.

In order to determine the composition of the internal atmosphere, DEVAUX cut a cylindrical hole into a potato and inserted a glass tube in it, which he tightened up with wax. He arranged the experiment in such a manner that out of this tube a little sample

TABLE 1.

Composition of the internal atmosphere of a winter-potato, stored at room-temperature.

Date	% CO_2	% O_2	% N_2
February 17	4.5	14.9	80.6
February 19	5.2	14.5	80.3
February 24	4.2	15.5	80.3
February 28	5.0	14.7	80.3
March 18	3.7	16.2	80.1
March 21	3.3	16.5	80.2
May 28	5.4	14.5	80.1
June 7	3.8	17.4	78.8
June 10	4.1	16.9	79.0
average	4.3	15.7	80.0

containing 0.1—0.2 cm^3 could be taken. It appeared, after one or two days, that equilibrium had been established between the gas in the intercellular spaces and the large hollow space formed by the hole and the glass tube.

He now analysed the gas in a large number of potatoes. Thus, he was able to follow during some time the alterations in the composition of the internal atmosphere of one and the same potato. We here give an example in figures derived from DEVAUX (Table 1).

The following conclusions can be drawn from this table: the internal atmosphere of a normal potato contains 3—5 % CO_2 and 14.5—17.5 % O_2. This proportion is maintained during storage, for a potato stored till October showed an internal atmosphere of the same composition, though it had heavily sprouted: i.e. CO_2: 6.6 %, O_2: 17.5 % and N_2: 75.9 %.

This method of analysing we did not meet with at all in the newer publications, though PFEFFER (6) and CZAPEK (1) do report the results.

Method of MAGNESS.

MAGNESS cuts, by means of a cork-borer, a cylindrical plug out of the potato. He puts it in an apparatus filled with mercury, which is connected with an open mercury-vessel by a tube at the lower edge. By lowering the mercury-vessel, a vacuum is formed in the apparatus. The vacuum causes the gas to escape from the potato. After the mercury-vessel is leveled, the gas can be transferred in a gas-analysing apparatus by means of a cock at the top of the apparatus.

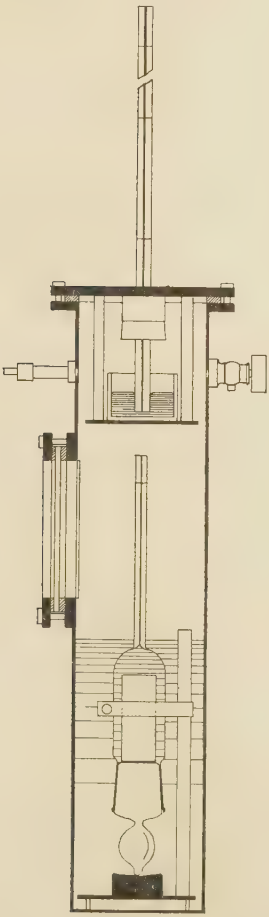


Fig. 1.

By this method MAGNESS determined the composition of the internal atmosphere of a potato stored at 22° like this:

CO_2: 34.4 % O_2: 5.7 % N_2: 59.9 %.

It is obvious at once that much more CO_2 and less N_2 are found than had been determined by DEVAUX. Other investigators working according to this method found a similar composition, e.g. SMITH:

stored dry: CO_2: 20.3 % O_2: 13.0 % N_2: 66.7 %
stored moist: CO_2: 30.6 % O_2: 7.0 % N_2: 62.4 %.

We could prove, in the first place, that the quantity of gas extracted from the potato in this manner is larger than the volume actually present in the intercellular spaces. In the second place, we could prove that the composition of the gas obtained depends entirely upon the method of extraction.

Determination of the volume of gas in the intercellular spaces.

Plugs of tissue, from which the volume had to be determined, were cut out of the potato with a cork-borer. The sample plug was pushed out into a glass vessel filled with water. The vessel was fitted at the lower edge with a glass stopper; at the top it ended in a capillary tube (see fig. 1). The glass vessel was shut under water in a brass cylinder in such a manner that the capillary tube above the surface of the water was visible through a looking-glass in the cylinder-wall. The whole apparatus was placed in a thermostate. When equilibrium between the temperature in the thermostate

and in the cylinder was established, the cylinder was shut by a cover provided with a manometer. The meniscus was read through a horizontal microscope, and thereafter air was pumped into the cylinder through a valve. The water in the potato, the water in the glass vessel and the air in the intercellular spaces were compressed. At several pressure the meniscus was read, so that the change in the volume of gas in the intercellular spaces could be determined at different pressures, the diameter of the capillary tube being known. A slight correction is made in view of the compression of the water, as known from an experiment without potato. The volume of gas in the intercellular spaces can be calculated, with the aid of the law of BOYLE from the formula: $V \times 76 = (V - \Delta V) \times P$. Table 2 gives an example of such a determination.

TABLE 2.

Determination of the volume of gas in the intercellular spaces of a cylindrical plug of potato weighing 15.5 g (fresh weight). Temp. 20°. Bar. pressure 76 cm. Diameter of capillary tube 1.75 mm².

Pressure P cm	Meniscus level mm	Correction mm	Removal meniscus mm	Change of volume ΔV mm³	Volume of interc. gas V mm³
76 + 0	0	—	—	—	—
+ 46.5	−25.9	0.8	−25.1	43.9	116
+ 72.8	−35.0	1.2	−33.8	59.2	121
+ 81.1	−37.1	1.4	−35.7	62.5	121
				average. .	119

This table shows that one and the same volume of gas is found at different pressures. It is proved to be gas, since it behaves according to the law of BOYLE. This method, therefore, is exact enough to determine the volume of gas in the intercellular spaces. It is remarkable that the former volume is not regained by lowering the pressure again. Probably the reason is that the compressed gas in the extremely small intercellular spaces is not able to overcome the elasticity of the rather solid cell-walls. This phenomenon cannot be caused by the dissolution of the gas in the intercellular fluid, as this gas should be already dissolved during compressing, and the law of BOYLE would then not hold good.

Determination of the volume of gas by the method of extraction.

The volume of gas in several potatoes was determined by extracting in vacuo according to the method of MAGNESS. The volume thus obtained was compared with the volume found by compression. After the extraction we again determined whether any gas remained. Table 3 shows the results of these experiments. It appears that always *more* gas is extracted than there is present in the intercellular spaces. Part of the gas extracted apparently comes from the fluid in the intercellular spaces or even in the cells.

Analysis of the gas extracted.

This analysis was performed with the manometric apparatus of VAN SLYKE, in such a manner that the method was somewhat altered, using quantities of 1 cm³ or less. To that purpose, instead of a 1 n sodium hydroxide solution, 0.1 n was used to absorb the CO_2, and a volume of 2 cm³ was read above the alkali-meniscus instead of a volume of 50 cm³ above the mercury-meniscus. Some analyses of laboratory-air proved the method to be exact enough. More details are found in PETERS and VAN SLYKE (5). Corrections of temperature were avoided by working at the time at the same temperature.

TABLE 3.

Comparison of the volume of intercellular gas determined by compression with the volume of gas extracted in vacuo. Volumes of gas in mm^3 per 10 g fresh weight. Temp. 20°.

Number of determination	Vol. of gas determined by compression		Extracted in vacuo		% of extracted gas gained from the gas phase
	Before extr.	After extr.	from gasphase	in total	
G 3	105	—	—	117	—
6*b*	—	22.5	—	212	—
6*c*	—	21	—	211	—
7*a*	102	39	63	299	21
7*b*	94	26	68	113	60
8*a*	77	25	52	156	33
8*b*	98	30	68	114	60
9*a*	98	3	95	100	95
9*b*	84	15	69	91	76

It appeared (table 4) that the gas extracted in vacuo contained too much CO_2 and rather little O_2; we obtained values in accordance with those of MAGNESS and SMITH. When the gas of the same potato was extracted with an under-pressure as slight as possible, the composition appeared to differ greatly. The lowest pressure at which gas escaped from the plugs of potato reached 50 cm under-pressure. But then, very little gas escaped, so that some more plugs had to be extracted in succession and the total amount analysed. In table 4 some analyses are given. We see from these that at 50 cm under pressure quite another composition is found than at extraction in vacuo.

TABLE 4.

Analysis of the gas in the intercellular spaces of cylindrical plugs of potatoes. Extracted at slight and strong under-pressure. In every experiment plugs of the same potato were used. Volume in mm^3 at 20°, calculated per 10 g fresh weight.

Number of experiment	Extraction in vacuo				Extraction at 50 cm under-pressure			
	$\%CO_2$	$\%O_2$	$\%N_2$	Vol.	$\%CO_2$	$\%O_2$	$\%N_2$	Vol.
W1	24.6	20.0	55.4	—	1.3	22.9	75.8	—
2	15.9	16.1	68.0	—	4.3	20.2	75.5	—
3	12.8	19.6	67.6	233	4.7	18.3	77.0	109
4	36.6	8.4	55.0	262	5.3	18.6	76.1	130
5	16.5	16.2	67.3	228	3.9	16.4	79.7	110

We also tried to obtain the gas from the internal atmosphere without extracting the dissolved gas, by infiltrating the plug with an organic liquid. With toluene we succeeded at an under-pressure of 25 cm in the extraction-apparatus. The gas obtained, however, could not be analysed, as the toluene vapours disturbed the determination. But at this slight under-pressure, too, more gas was extracted than was present in the gasphase of the intercellular spaces, as is shown by the following figures:

volume of gas before extraction: 36.5 mm^3 per 10 g fresh weight
,, ,, ,, after ,, : 2.8 mm^3 ,, 10 g ,, ,,
volume of gas extracted : 55.5 mm^3 ,, 10 g ,, ,,

When plugs of potato were extracted at 40 cm under-pressure in the extraction-

apparatus, in water freed from air, the gas obtained could be analysed. This method has this advantage, that the same plug of potato can be extracted, first at 40 cm under-pressure and then in vacuo. Table 5 gives the analyses of the gas thus obtained. It is obvious that the composition of the gas successively extracted is quite different.

TABLE 5.

Composition of the gas in the intercellular spaces of plugs of potato, obtained by extraction under water at 40 cm under-pressure and in vacuo. Temp. 20°. Volume of gas calculated in mm^3 per 10 g fresh weight.

Number of determination	Extracted under water				Extracted in vacuo			
	% CO_2	% O_2	% N_2	Vol.	% CO_2	% O_2	% N_2	Vol.
G11	0.4	17.4	82.2	25	31.1	6.3	62.6	130
12	0.3	12.1	87.6	61	18.0	4.3	77.7	137

It may be admitted that the gas, which escaped at a slight under-pressure, consists for a large part of the real intercellular gas, and that, on the other hand, the gas thereafter obtained in vacuo consists of dissolved gas. The much higher percentage of CO_2 can be explained by the greater dissolubility of the CO_2 than of the other gases. We will not say, however, that the gas obtained by extraction at a slight under-pressure under water, or by extraction at a somewhat greater under-pressure under mercury, shows the right composition.

The composition of the gas obtained in vacuo can not be deduced from the sum of the gas in the gas-phase and the gas dissolved in the intercellular liquid, as the short time of extraction does not give sufficient opportunity to the gas to establish equilibrium with the large gas-room on the outside of the potato, supposing that equilibrium can be still established in a damaged potato. The CO_2 may also be bound in the cells themselves but we know nothing about this with any certainty.

In the old method of DEVAUX the potato is damaged only slightly, and for days the intercellular gas has an opportunity to establish equilibrium with the gas space from which a sample can be taken. This method, therefore, gives better results than the method of extraction. The results, too, are more trustworthy in our opinion. DEVAUX also pointed out that the composition of the gas of several plants was the same during the same period of vegetation. It is obvious that he found about the same value as we found when extracting at a slight under-pressure (table 4). It is remarkable, too, that DEVAUX observed that the composition of the gas during storage of the potatoes did not alter until sprouting, although SMITH (who worked with the extraction-method) found that the percentage of CO_2 during storage did increase, and was, indeed always more than DEVAUX observed. Probably the alterations observed are caused to a large extent by the respiration of the potato, so that an increasing quantity of CO_2 is extracted from the intercellular liquid. The alterations in the composition of the extracted gas mean only very little, and any conclusion would be premature.

Summary.

The volume of gas in the intercellular spaces of potatoes was determined, and experiments showed that the quantity of gas that can be extracted by the method of MAGNESS (4) is always more than the real volume present. Moreover it appeared that, with different methods of extracting, quite different results were obtained, probably because gases from the intercellular liquid were also extracted. The method of extracting, therefore, does not give good results, and it is risky to use this method to determine the composition of the intercellular gas. It was shown that the old method of DEVAUX (3) is preferable for the determination of this gas.

We are much indebted to Dr A. W. H. VAN HERK for his valuable help and criticism during the course of this work.

LITERATURE.

1. F. CZAPEK, Biochemie der Pflanze, 2. Aufl. **3**, 23 (1921).
2. W. B. DAVIS, Bot. Gaz. **81**, 323 (1926).
3. H. DEVAUX, Ann. Sciences natur. 7. Sér. **14**, 397 (1891).
   ————, Bull. Soc. bot. France **37**, 272 (1890).
4. J. R. MAGNESS, Bot. Gaz. **70**, 308 (1920).
5. J. P. PETERS and D. D. VAN SLYKE, Quant. Clinical Chem. **2**, 106 (1932).
6. W. PFEFFER, Pflanzenphysiologie, **1**, 181 (1897).
7. O. SMITH, Hilgardia, **4**, 274 (1929).
8. N. C. THORNTON, Contrib. Boyce Thompson Inst. **10**, 339 (1939).

Amsterdam, Laboratory of Plant Physiology of the University.

Botany. — *Some technical remarks about the colchicine treatment of Gramineae.* By C. VAN DILLEWIJN. (Communicated by Prof. A. H. BLAAUW.)

(Communicated at the meeting of October 25, 1941.)

In surveying the results hitherto obtained with colchicine treatment in various plant-families one is struck by the fact that the share of the Gramineae is comparatively small. This is all the more striking, because many of our most important economic plants belong to this family. It may therefore be safely assumed that many attempts have been made to induce polyploidy in our cereals by means of colchicine and that the relatively slight success is due to peculiarities that are typical for this family.

Which are these peculiarities? In order to answer this question it should be remembered that with Gramineae the general complaint is that seedtreatment with colchicine solution actually calls up typical polyploidy phenomena in the plumula, but that the roots generally die soon after the treatment. If the concentration of the colchicine solution and the duration of the treatment are chosen in such a way that the roots do not die, this treatment proves to be insufficient for causing lasting polyploidy in the vegetation-point of the plumula.

The question narrows down to this, therefore, that a certain colchicine treatment causes much stronger reactions in the radicula than in the plumula. And this may logically be brought into connection with peculiarities in the structure of germinating cereal grains: the almost naked radicula, affected directly by a colchicine solution, and the vegetation point of the plumula surrounded by young leaves and a coleoptile to which vegetation-point the solution can hardly or not at all penetrate.

On the basis of these considerations means were sought

a. to weaken the effect of colchicine on the radicula;

b. to strengthen the effect of colchicine on the vegetation-point of the plumula;

c. to make a combination of *a.* and *b.*

The first experiments were made in 1940 with rice, viz. two Italian varieties, BERTONE and ALLORIO. If the seeds are allowed to germinate *under* water, first the plumula is developed and for a time the radicula does not appear. If at this stage the two coleoptiles and the young leaves in them are cut off closely above the vegetation-point and the seedlings thus treated are immersed in a colchicine solution, it may be expected that the radicula will be ill accessible to this soluton, the vegetation-point of the plumula on the other hand will be easily accessible to it. In this way a large percentage of mutants can easily be produced (cp. Landbouwkundig Tijdschrift 53, 544—545, 1941), the prescription for this treatment being as follows:

> Sterilize the kernels for ½ hour in a ¼ % Germisan solution. Allow the treated kernels to germinate under water in a thermostat at 27° C during 6 days. Cut off the coleoptiles and young leaves until about 1 mm remains. Keep the thus treated seedlings for 48 hours in a thermostat at 31° C in a solution of 0.1 % colchicine.

Seedlings treated in this way give both with BERTONE and with ALLORIO mutants which clearly deviate from the normal type: the plants are shorter and more compact, the colour is darker, the leaves remain green for a longer time and the ripening takes place much later; tillering continues almost unlimited, the stomata are larger, the ligula is longer, the caryopsis is clearly larger with smaller length/breadth index; further the lemma is more coarsely hairy and has a strikingly long awn (in normal plants the awns being almost wholly absent). The mutants are fairly fertile, annually forming plenty of seed and this seed is in germinating power hardly inferior to the seed of normal plants. The accompanying plate shows a normal plant and a mutant of ALLORIO.

Foto: C. J. VAN DE PEPPEL.

ALLORIO rice. Left normal plant, right mutant.

Owing to this favourable result similar experiments were made in 1941 with rye, wheat, barley and oats.

The suppressing of the root development, although possible, is a laborious and cumbrous process for these plants, so that it was abandoned. If grains are allowed to germinate in moist conditions until the coleoptile has a length of 2—6 mm and if then the coleoptile and the enclosed leaves are cut off to a length of 1—2 mm above the vegetation-point, such seedlings, after immersion in colchicine solutions of 0.01—0.3 % for 16—43 hours, show the well-known polyploidy phenomena in the plumula and the roots die off.

In order to weaken the effect of the colchicine solution on the roots it was attempted to shorten the treatment of the roots without, however, noticeably decreasing the treatment of the plumula. For this purpose the coleoptile of seedlings with a coleoptile-length of about 4 mm was cut off at half its length, so that there remained part of a coleoptile of 2 mm with enclosed young leaves. The seedlings thus treated were immersed in bottles with colchicine solutions of 0, 0.01, 0.03, 0.1, 0.3 and 1 % and the bottles were placed under the bellglass of an air-pump in which the air was alternately pumped away and admitted. After a quarter of an hour the seedlings were taken out of the bottles and the roots were dipped a few times in water in order to wash away the colchicine solution attached to them. Then the seedlings were planted in moist sand. At the moment of planting therefore the radicula had for about a quarter of an hour been in contact with the colchicine solution; the same holds good for the plumula, but besides, the coleoptile cylinder was at the moment of planting still filled with colchicine solution which continued for a considerable time to have its effect on the vegetation-point of the plumula. In this way the colchicine solution is made to affect the plumula much longer than the radicula. In order to remove the decomposing effect of the light on the colchicine it may be recommended to preserve the seedlings in a dark room during the first 48 hours after planting.

By way of example the result of such an experiment with rye (PETKUSER summerrye) is given here.

Two weeks after the treatment the situation was as follows:

Series 1 %: Plumula clearly globularly swollen; good root development.
Series 0.3 %: In part of the plants plumula globularly swollen, in the rest thick cylindrical; good root development.
Series 0.1 %: Plumula predominantly egg-shaped; good root development.
Series 0.03 %: In half of the plants the primary leaf has broken through the coleoptile and has a length of 2—5 cm, in the rest the plumula is thickened egg-shaped; good root development.
Series 0.01 %: In all plants the primary leaf has broken through the coleoptile and has a length of 3—6 cm; good root development.
Series 0 %: Like the preceeding series.

It is remarkable that the well-known polyploidy phenomena could be caused in the plumula, without at the same time killing the roots.

Five days later (so 19 days after the treatment) the situation in this experiment was as follows:

Series 1 %: Plumula dead; roots in the majority of cases soundly developed.
Series 0.3 %: Like the preceeding series.
Series 0.1 %: Plumula mostly dead; roots in the majority of cases soundly developed.
Series 0.03 %: Plumula mostly living; primary leaf 3—10 cm; sound root-system.
Series 0.01 %: Plumula and root-system normally developed.
Series 0 %: Like the preceeding series.

In one respect this result is interesting. The concentrations of 1 and 0.3 % appear to have been lethal for the plumula but not for the roots. At least in the (dissimilar)

treatment applied here. This is exactly the opposite of the result as a rule obtained hitherto in colchicine treatment of cereal seeds, viz. that the roots died off and the plumula remained living.

It is clear that between these two extremes there must be a range where both the plumula and the roots react favourably on a colchicine treatment.

Another possibility of exposing the vegetation-point of the plumula to the effect of colchicine, without damaging the roots, is the following.

The seeds are allowed to germinate in moist sand in a dark room, until the coleoptile has reached a length of 40—50 mm. Then an incision at one side is made in the coleoptile 5—10 mm below the top and then the top is broken off. The primary leaf now projects above the coleoptile cylinder and when it is carefully pulled it breaks off at the base, so that it can be removed as a whole.

The empty coleoptile cylinder can be filled with a colchicine solution by means of a glass capillary; if then the plants are placed in the dark for 48 hours, so that the decomposing effect of the light is eliminated, the vegetation-point is exposed to the effect of the colchicine solution and the root-system, which goes on growing normally in the soil, remains entirely intact.

Such a treatment was applied to rye (PETKUSER zomerrogge), barley (MANSHOLT's tweerijige zomergerst), oats (ADELAAR haver) and wheat (MANSHOLT's witte zomertarwe) and various concentrations of colchicine were compared. The best results were obtained with concentrations between 0.1 and 0.5 %, dependent on the varieties investigated.

Instead of decapitating the coleoptiles one may also wait till they are broken through by the primary leaf. In many cases the groove which then arises at the side of the top more easily admits a fine glass capillary than the opening that arises when the coleoptile is decapitated.

In many cases it may be recommended to place the plants a few times under the bell-glass of an airpump after the coleoptile cylinders have been filled with colchicine solution, in order to allow the solution to penetrate as far as the vegetation-point.

Wageningen, October 1941.

Botany. — *The growth of the petioles of waterplants in solutions of phytohormones.* (From the Botanical Institute, Government University, Leyden.) I. By G. L. FUNKE. (Communicated by Prof. L. G. M. BAAS BECKING.)

(Communicated at the meeting of October 25, 1941.)

I. *Introduction.*

My experiments on waterplants have shown that the addition of phytohormones to the water has, on the whole, little effect on the acceleration of growth and provokes abnormal phenomena as long as the temperature is below 20° (5,6). At higher temperatures, on the other hand, their influence appears to be enormous (7,8). Up to now I have mainly used a concentration of 3 mg/L, because this was one of the optimal concentrations found by VEGIS (16) in his experiments with heteroauxin on the premature sprouting of the turiones of *Stratiotes aloides*; no other authors, so far as I know, have ever applied growth substances on waterplants. This year I studied the effect of lower concentrations of different kinds of phytohormones on the petioles of several species of *Nymphaeaceae* and other plants.

Part of the plant material was provided by the courtesy of the Zoological Gardens at The Hague and the Botanical Gardens of the University of Amsterdam. Experiments were made with seedlings and with specimens grown from tubers. Four basins were used in the Victoria hothouse of the Botanical Gardens at Leyden, especially built for the observation of growth in deeper waterlayers; two of them have a depth of 150 cm, two of 200 cm; during hot summerdays the temperature of the water varied between 26° and 31° at the surface, between 24° and 28° at the bottom; during the rainy period in August it sank gradually to 23°—20°. Other experiments were made in small aquariums of 20 cm deep; these were placed on radiators in a tropical hothouse; their temperature varied between 24° and 29°. Towards the end of the summer I could make some experiments in an aquarium of 40 cm deep which was electrically heated at a constant temperature of 28°. Only three kinds of hormones have been applied, viz. heteroauxin, beta-indole-butyric acid and alpha-naphtalene acetic acid, beta-indole propionic acid and beta-naphtalene acetic acid having been shown to have only a slight influence or no influence at all (6).

II. *The influence of heteroauxin, beta-indole butyric acid and alpha-naphtalene acetic acid.*

Four equally developed young specimens of *Nymphaea amazonum,* grown from tubers, were planted in aquariums of 20 cm deep. Part of the leaves were full grown, which means that their blades floated on the surface and that their petioles elongated no more. A few days later the hormones were added in a dose of 0.5 mg/L; one aquarium served as control. The petioles were measured every day; they showed facts which have appeared invariably in all my former experiments. The older leaves reacted very feebly or not at all; the very young ones needed some days before responding to the addition of the hormones; the fresh adult ones, ranging between, did not react equally well, apparently owing to a difference of their respective ages, although their appearance was very much alike. Table 1, therefore, gives only a selection of the measurements which illustrate best the action of the hormones.

Part of the control leaves showed a renewal of growth towards the end of the experiment, but their elongation, nevertheless, stayed very much behind that in the solutions of growth substances. As the depth of the aquaria is only 20 cm, the result was that the greater part of the petioles floated on the surface in large coils which intertwined.

We see that alpha-naphtalene acetic acid is by far the most active hormone and heteroauxin the least. In naphtalene acetic acid the older leaves were strongly twisted

TABLE 1. Petioles of Nymphaea amazonum in solutions of different hormones (petioles in cm; depth of watercolumn 20 cm).

Hormones	May							June		
	24	26	27	28	29	30	31	2	3	4
Control	23	24	24	24	25	25	25	25	25	25
	13	20	22	23	25	30	35	42	45	46
Heteroauxin	25	66	70	71	76	78	81	82	84	85
0.5 mg/L	18	47	50	54	56	60	63	66	67	67
β-ind. butyr. acid	24	63	77	84	89	92	93	100	101	101
0.5 mg/L	20	58	77	85	91	99	103	105	110	110
α-napht. acet. acid	21	73	100	114	121	129	132	134	137	138
0.5 mg/L	8	29	41	59	85	108	117	136	142	149

and fragile and often broke in 2 or 3 pieces when they were measured; this seems to indicate that the concentration 0.5 mg/L is more or less noxious; we will see further on that the cell measurements confirm this view, cell elongation taking the upperhand of cell division.

This experiment has been repeated with identical results; it has been made also with the species *N. Boucheana* and *N. jubilé lilacina*. In every case we see that the action of alpha-naphtalene acetic acid is most efficient and also slightly noxious, witness the twisting of the petioles, that of heteroauxin least, while that of beta-indole butyric acid is intermediate. The definite lengths were always reached in 10 à 14 days.

Experiments with seedlings of *N. Obergaertner, N. Devoniensis, N. zanzibariensis rosea, N. jubilé lilacina* and *N. stellata* were made under the same conditions. As the plants were very young the concentration used was only 0.25 mg/L and even this dose proved to be rather too much; nevertheless, all these species clearly demonstrated the same succession in effect of the hormones in furthering growth. The same holds true for *Victoria regia* (first floating leaves) and petioles of *Ranunculus sceleratus*. Only one species showed about the same rate of growth in everyone of the three hormones, viz. *Limnanthemum nymphaeoides*, although also here it appeared that the higher concentrations of alpha-naphtalene acetic acid were more noxious than those of heteroauxin and beta-indole butyric acid. A detailed account of the experiments with the latter species has been given elsewhere (9). Full data of the measurements of all species investigated are at the disposal of everyone interested; as they only confirm those given in table 1, it is needless to give them here.

III. *The influence of different concentrations of alpha-naphtalene acetic acid.*

As it has been stated, that of the three hormones used, alpha-naphtalene acetic acid has by far the strongest furthering effect on the growth of the petioles of many waterplants, this substance was consequently applied in various concentrations.

Four young specimens of *Nymphaea amazonum* were planted in the same way as described above and the following doses were added: 0, $\frac{1}{4}$, $\frac{1}{2}$ and 1 mg/L. Table 2 gives an idea of the rates of growth.

We see that the utmost lengths reached in the solutions $\frac{1}{4}$, $\frac{1}{2}$ (table 1) and 1 mg/L do not vary noticeably; yet it may not be said that the plant is insensitive to the difference in concentrations. As has been said above, 0.5 mg/L is already slightly noxious which is indicated by the fragility and the twisting of the petioles and their very long cells; these phenomena all occurred in the 1 mg/L solution also and to a higher degree; the greater

part of the petioles did not reach the ultimate length because they broke long before the end of the experiment while being measured. Therefore I consider the concentration

TABLE 2. Petioles of Nymphaea amazonum in different concentrations of α-naphtalene acetic acid (petioles in cm; depth of watercolumn 20 cm).

Concentrations	June										
	4	5	6	7	9	10	11	12	16	19	23
Control	16	20	22	24	25	27	28	28	40	44	44
	2	4	6	9	17	20	21	21	25	25	36
$\frac{1}{8}$ mg/L	22	34	53	71	86	97	100	103	104	109	109
	8	18	29	41	71	83	94	97	104	111	111
$\frac{1}{4}$ mg/L	20	36	60	79	112	113	120	127	134	134	134
	1	1	5	9	20	46	63	76	146	163	170
1 mg/L	19	32	58	94	116	130	133	134	136	139	139
	13	21	43	68	124	138	150	159	165	166	166

0.25 mg/L as the optimal one. Experiments with higher concentrations could not be made in a sufficient number for two reasons: (1) the plant material was restricted; this holds true for the specimens grown from tubers, (which will be the difficulty in every botanic garden); seedlings were available in a larger number but we will see further on that they are less suitable for this sort of experiment; (2) in the course of the summer the plants became too tall to be planted in small aquaria of 20 cm depth; the experiments had to be continued in the large basins in the Victoria hothouse, but as these have a capacity of 1500 L each, it will be easily understood that concentrations of more than 1 mg/L could hardly be applied.

The few experiments, however, which I could make, and those of former years, all confirm the impression that concentrations higher than 1 mg/L of all three growth-substances are still more noxious and check the longitudinal growth. One of these may be briefly described. *N. jubilé lilacina* is a species whose responses to phytohormones are in all respects very similar to those of *N. amazonum*; four young specimens were planted in aquariums of 20 cm deep and 3 mg/L of the three hormones were added. Growth in heteroauxin and beta-indole butyric acid was only a little more rapid than in the control, but in alpha-naphtalene acetic acid it was nearly equal to that. All sorts of abnormal phenomena, observed during my earlier investigations with this concentration, occurred again and most distinctly in alpha-naphtalene acetic acid: strong twisting of the petioles, stiff hyponastic curling of the blades which, moreover, remained very small. After 7 days the plants were washed and removed into pure water; they recovered within 48 hours, and most of them even within 24 h.; the twisting of the petioles, the curling-up of the leafblades disappeared and the blades grew out to normal dimensions; the petioles did not elongate any more (except of course the very young ones). Seven days later the hormones were again added, but this time only in a dose of 0.5 mg/L. The same leaves, which had reacted so unfavourably to the concentrations 3 mg/L, started growth anew (except in the control aquarium) and at the same pace as has been observed in all similar experiments. After 10 days they had reached lengths of 28 à 32 cm in control, 65 à 70 cm in heteroauxin, 70 à 80 cm in beta-indole butyric acid and 130 à 146 cm in alpha-naphtalene acetic acid. Therefore we may conclude that the hormones are distinctly noxious in a concentration of 3 mg/L and especially so alpha-naphtalene acetic acid which has obviously a lower optimal concentration than the other two. My experiments of 1938 (6) and 1939 (7) confirm this view (with the restriction of *N. Boucheana*, which form is apparently less sensitive to higher concentrations).

The experiments in the deep basins took place as follows: young plants, which had

developed in a shallow layer of water, were planted in pots and these were hung in the basins at a depth somewhat greater than the length of the petioles; next day the leafblades had reached the surface and the pots were lowered 20 à 30 cm; this was repeated every day and each time part of the leafblades (not the oldest ones) appeared at the surface well within 24 hours, so that at the end of 4 to 6 days the pots had reached the bottom of the basins and even then growth did not stop but continued for at least some decimeters. The leaves made a very healthy impression and nothing indicated that they were at the end of their achievement; (the pots having a height of 20 cm, the real waterlayers were resp. 130 and 180 cm). The very youngest leaves, which only begin to develop after the immersion is completed, do not, as a rule, reach the surface but if they do, with "visible exertion" and after a considerable time.

Adult petioles of ± 30 cm thus elongate up to ± 200 cm in about one week. This is in itself a remarkable performance which, however, I rather expected after what I have observed in former years. When nothing further is done to them the leaves do not elongate any more, their "excess growth" (by which I mean the part of the petiole above the surface) is restricted to 30 à 40 cm. When, however, 0.25 mg/L alpha-naphtalene acetic acid is added, growth starts anew and during 8 à 12 days elongation continues at a very rapid pace. The petioles reach lengths of more than 300 cm, which means that their greater parts are meandering on the surface, intertwining with each other. A few examples out of many others may give an impression of the rate of growth in the deep basins after the surface has been reached.

TABLE 3. Excess growth of petioles of Nymphaea amazonum in watercolumns of 150 and 200 cm (petioles in cm).

	No.	June											
		16	17	18	19	20	21	23	24	25	26	27	28
Control													
Basin 150 cm	1	142	143	150	156	156	157	160	162	163	163	164	164
Basin 200 cm	2	192	193	200	206	206	207	210	212	213	213	213	213
α-napht. ac. acid													
0.25 mg/L	1	140	167	175	195	207	217	231	236	236	236		
Basin 150 cm	2	132	150	185	203	230	238	257	262	270		270	
	3				100	160	218	270	280	288	290	296	296
	4					58	94	224	263	290	310	310	310
	5										108	160	202

Leaves No. 1 and 2 in the solution of naphtalene acetic acid are obviously rather old ones which are no more capable of extreme responses to the hormone, although an elongation from 30 to resp. 140 and 132 cm as a result of the deepening of the water-column, followed by the growth since June 16th as indicated in the table, remains remarkable all the same. Nos. 3, 4 and 5, however, were younger ones which were observed just at the moment of their utmost capacity of reaction. The strongest elongation in 24 h equals and even surpasses the record of growth rate which I observed in 1940 (7) in a 3 mg/L solution of beta-indole butyric acid with *N. Boucheana* and other species. No. 4 has grown 130 cm from Saturday 21/6 till Monday 23/6; I estimate that in 24 h the increment has been resp. ± 70 and ± 60 cm; we will see farther on that even this record can be beaten.

This experiment was repeated several times with *N. amazonum* and once with *N. jubilé lilacina, Bouchcana* and *Lotus rosea*; it always yielded identical results. During August and September the weather was unfavorable and the temperature of the water sank to 23° and even to 20°; this retarded the growth, but all the same the ultimate lengths

reached were about 300 cm. Daily growth of 60 cm and more did not occur very often, but 40—50 cm per 24 h were quite usual.

I applied several concentrations of alpha-naphtalene acetic acid in these basins also, viz. $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$ mg/L. There being only four deep basins, the experiment had to be made in two successive parts. The pots were not lowered deeper than 150 cm. Table 4 gives the data of the full grown petioles; those which could not reach the surface were neglected.

TABLE 4. Ultimate lengths of petioles of Nymphaea amazonum in a watercolumn of 150 cm and in different concentrations of α-naphtalene acetic acid.

	Length of petioles in cm							Average	Average of excess growth
	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6	No. 7		
Control	202	183	178	152	152	140		168	38
α-napht. ac. acid $\frac{1}{64}$ mg/L	186	179	179	178	166	149		173	43
α-napht. ac. acid $\frac{1}{32}$ mg/L	255	240	196	184	162	146		197	67
α-napht. ac. acid $\frac{1}{16}$ mg/L	233	225	162	159	153			186	56
α-napht. ac. acid $\frac{1}{8}$ mg/L	300	235	219	170	152	151		205	75
α-napht. ac. acid $\frac{1}{4}$ mg/L	310	295	283	275	267	250	230	273	143

The specimen in the $\frac{1}{16}$ concentration was a less prosperous one than the others, hence its rather poor development. The number of the petioles is by far too small to take an average from them; it will e.g. be seen that in the controlbasin one grew to a length which is not reached in the $\frac{1}{64}$ solution. Given due consideration to all this it is surprising that the averages nevertheless indicate to some degree a relation between growth and concentration of the hormone. These results seem to confirm my opinion that 0.25 mg/L alpha-naphtalene acetic acid (2.5×10^{-7} or $\pm$ 7×10^{-5} mol) is optimal for exerting the utmost of growth capacity in the petioles of *N. amazonum* and probably of many more species of *Nymphaeaceae*. It should be remembered, however, that in $\frac{1}{8}$ and even in $\frac{1}{32}$ mg/L (3.1×10^{-8} or $\pm$ 6×10^{-6} mol) individual petioles can show the same rate of elongation.

A similar experiment has been made with *N. Boucheana*; as, however, my material of this species was scanty, I had to be content to test it in three concentrations only, viz. $\frac{1}{32}$, $\frac{1}{16}$ and $\frac{1}{8}$ mg/L. The average excess growth in the control was 36 cm, in $\frac{1}{32}$ mg/L 57 cm, in $\frac{1}{16}$ mg/L 70 cm and in $\frac{1}{8}$ mg/L 109 cm. Also in this case there seems to exist a relation between growth and concentration of naphtalene acetic acid. These results, added to those obtained in the small aquaria indicate that for this species too the optimal concentration must be about 0.25 mg/L.

N. amazonum and *N. Lotus rosea* have been tested in the deep basins in two concentrations of beta-indole butyric acid, viz. $\frac{1}{32}$ and $\frac{1}{4}$ mg/L; their excess growth was equal to that in the control. This confirms my former experiments (and those of 1940); for optimal growth achievements in this substance higher concentrations are needed.

IV. *The two main factors inducing rapid elongation.*

There are two methods to induce a sudden vigorous renewal of growth of adult petioles of waterplants: (1) transplanting in a deep waterlayer and (2) adding of phytohormones.

In the last described experiments these methods have been applied in succession and each of them has shown its action very neatly: (immersion in a deep waterlayer in which

growth substance had already been dissolved gives the same results; adding of growth substance first and deepening of the waterlayer afterwards has not yet been tried.)

The question which of these methods has the strongest influence is not simple to answer, because it depends on many factors of which the most important is the condition of the plant material. Seedlings seemed to be preferable to specimens grown from tubers because they present a more homogeneous material. It appeared, however, that in their very juvenile stage they are not suitable for experiments; they hardly responded to a deepening of the waterlayer and even very low concentrations of phytohormones were apt to injure them. Only towards the end of July (they were sown in April) they were sufficiently advanced to give reliable results; in most cases this stage is not characterised by morphological changes but must depend on metabolical processes. The plants remained throughout the summer in shallow dishes in a few cm of water and developed 5 or 6 leaves per specimen with petioles of not much more than 10 cm. As soon as they appeared sufficiently matured a number of experiments were made with them; they were planted in an aquarium at depths of 8, 18, 28 and 38 cm, at several levels in the deep basins, in pure water, and in various concentrations of growth substances. They showed that, although elongation was sometimes considerable, yet their growth capacity had by far not reached the amazing potentialities which are to be seen in older material. In the deep basins only those specimens were able to reach the surface which had already formed small tubers. This was the case with *N. zanzibariensis rosea*; the petioles elongated in 18 days from about 20 cm to more than 200; this is quite an achievement for these slender organs and I think it most remarkable that after this the small tubers not only were still present, but that their volume had even slightly increased; one would expect quite the contrary; this opens possibilities for further research. Subsequent addition of growth substances could not induce any more growth, nor could it stimulate those petioles which had stayed behind (compare (9), *Limnanthemum nymphaeoides*).

It seems that the height of the watercolumn is the most efficient factor for inducing rapid longitudinal growth in very young plants; as they grow older hormones gradually become equally important, till later on they take the upperhand and finally can be the chief means of attaining extraordinary dimensions. This is further demonstrated by an experiment which I made towards the end of the summer in an aquarium of 40 cm deep; it lasted from August 27th until September 18th and was made with material grown from tubers which had developed throughout the summer in shallow water. It was therefore some three months older than that which I used in the aquaria of 20 cm deep; all traces of

TABLE 5. Petioles of Nymphaeaceae in 0.25 mg/L alpha-naphtalene acetic acid (petioles in cm; depth of watercolumn 40 cm).

Species	No.	September								
		3	4	5	6	8	12	16	17	18
N. amazonum	1	41	64	90	108	115	115			
	2	40	91	136	183	220	220	222	228	236
	3		19	44	92	252	350	367	370	
	4						35	212	222	222
N. stellata	1	50	56	60	77	89	95	101	105	
	2	48	73	97	105	135	142	145	145	
	3	32	61	108	125	141	172	172	178	
	4			4	9	37	130	210	210	
N. Devoniensis	1	47	58	68	100	110	116			
	2	43	68	102	118	130	151	158	160	
	3	40	89	134	165	195	203	213	213	

primary leaves had long since disappeared, the leaves were more robust with large blades and thick petioles. Three species were used, viz. *N. amazonum, N. stellata* and *N. Devoniensis.* Their longest petioles measured up to 40 à 45 cm; during one week they did not change and then, on September 3d, 0.25 mg/L alpha-naphtalene acetic acid was added. Once more I had the occasion to observe the almost explosive renewal of growth. Some data will illustrate this in table 5.

No. 1 of all three species are typical samples of leaves which, although looking quite fresh, have just passed the age on which the "top response" is possible. No. 3 of *amazonum* and No. 4 of *stellata,* on the other hand, are very young leaves which show clearly that growth is never extraordinarily rapid in the beginning, but that at a certain stage it can take an enormous jump. This is especially the case in No. 3 of *amazonum* which has beaten two records: (1) 160 cm growth in 48 hours (probably resp. ± 90 and ± 70 per day) and (2) an ultimate length of 370 cm which has not even been reached in the deep basins.

It is perhaps not superfluous to emphasize that in an experiment like this one it is indispensable to distinguish the leaves by individual characteristics in order to identify them from the nearly undisentangible mass of intertwined petioles and that utmost care is needed to undo them from it without breaking them.

(Continued.)

Botany. — *The influence of oxygen-nitrogen mixtures upon the dwarfing of Ardisia Crispa (Thunb.) A. DC.* By Miss R. BOK. (From the Botanical Institute, University of Leyden.) (Communicated by Prof. L. G. M. BAAS BECKING.)

(Communicated at the meeting of October 25, 1941.)

Ardisia crispa, a tropical shrub belonging to the Myrsinaceae, shows a curious symbiosis with an aerobic bacterium, Bacterium foliicola Miehe, as first described by MIEHE. In the doctor's thesis of PH. DE JONGH a tentative explanation as to the nature of this symbiosis is given; the bacterium, covering the vegetation point with a film, acts as an "oxygen-trap" and lowers the oxygen-tension in the meristematic region of the stem. Ardisia plants without bacteria show a stunted development, the resulting dwarf, which was called a "cripple" by DE JONGH, might therefore perhaps be a product of an overdose of oxygen at the vegetation-point.

It seemed logical to try to cultivate bacteria-free Ardisia plants in an atmosphere with diminished oxygen-tension in order to see whether they would develop into normal plants.

The older cripple, however, is mutilated too much to be able to revert to the normal, while the tissue of young plants showed more promise in this direction.

It seemed, therefore, advisable, to use sterile seeds, in which the symbiont was killed by heating to 40° C. for 48 hours. This internal sterilisation was followed by external sterilisation. First the seed was immersed for 2 minutes in 80 % alcohol to remove the wax, after which the seed was treated with 1 % calciumchlorohypochlorite for 30 minutes. After a final rinsing with sterile water the seed was transferred to the culture-flask by means of a sterile needle. Germination took place in sterile environment in an oxygen-nitrogen atmosphere.

This method has a drawback in that the initial germination and also the root-formation takes place in abnormal atmosphere.

Normal, internally-non-sterile seeds were used as controls. In order to keep the composition of the gas-mixture constant, independent of metabolism and of gas-exchange with the outer atmosphere, a closed circuit was constructed.

The following gas-mixtures were used:

I.	100 %	N_2		
II.	95 %	N_2	5 %	O_2
III.	85 %	N_2	15 %	O_2
IV.	78 %	N_2	22 %	O_2
V.	40 %	N_2	60 %	O_2
VI.			100 %	O_2

For each mixture two carboys of 15 litres capacity were used. (See Figure: A and B.)

Carboy A was filled with a highly dilute solution of $HgCl_2$ to prevent algal growth.

This solution was previously saturated with the gas-mixture. At the beginning of the experiment carboy B (calibrated) was filled to the zero-level with the solution of $HgCl_2$ and 15 litres of the gasmixture. Through funnel *a* sublimate-solution was streaming up to outlet-opening of the vacuum-tube *b*. The continuous outflow of water from A forced the gas-mixture through the train, consisting of bottle *d*, filled with concentrated H_2SO_4 in order to dry the gas previous of its passing through two parallel circuits e_1 and e_2, each consisting of a flow-meter of a type used by WILSON and GEORGI.

The capillary I was interpolated in order to prevent extreme reactions in the manometer *m*. Capillary III served the same purpose. The velocity of the gas was regulated by varying the bore and the length of capillary II.

Cylinder *f* serves to sterilize the gas by means of its passing through sterilized cotton. Via tube *g* the gas reached finally the culture-flask *h*. After passing over the culture medium it left the train through a cotton-filled tube *k* and a bubble-counter *n*.

The culture-flask was made of Jena glass (model FERNBACH conical), contents 1900 cc. Due to the large bottom area, 16 seeds could be placed on about ½ litre of culture medium.

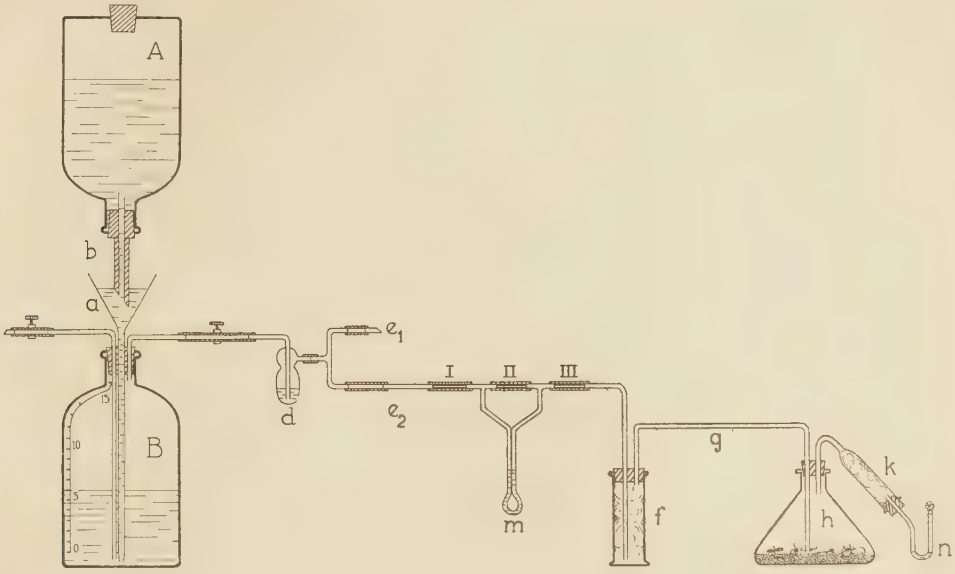


Fig. 1.

Seeds were used of three mother-plants, cultivated in a hot house in the Botanical Gardens.

The gases were mixed in the tonometer in the usual way, no correction was made for admixtures.

The culture medium was rainwater to which was added 1 % of purified agar and a modified KNOP-mixture, consisting of

$Ca(NO_3)_2$	4.0 grammes per 7000 cc water
KNO_3	1.0
$MgSO_4 . 7$ aq.	1.0
KCl	0.5
$FeCl_2$	trace
$MgNH_4PO_4$	1.0,

the latter salt being used because of its superior buffer-action as compared with KH_2PO_4. The chemicals were the C.P brand of the firm of BROCADES. The salts were dissolved separately, and the five solutions mixed.

The apparatus was partly autoclaved, partly oven-heated prior to use. The assembling took place in a formalin-sterilized cubicle after which the entire apparatus was mounted in a greenhouse which was thermostat-controlled at ± 20° C. The plants were shaded from direct light.

Twelve tonometers were used, each serving two culture-flasks, one containing 16 normal seeds, the other 16 sterile seeds. Every series was duplicated and six gas-mixtures were used.

The experiments started June 1939 and were kept going until November 1939. The plants were kept on the same substrate, but in atmospheric air, until February 1940.

A blunt vegetation-point is a certain sign of the crippled nature of the plant. This was therefore used as a criterion.

In pure N_2 or *in 95 %* N_2 germination was almost absent, and leaf-formation did not take place in the few seeds that showed signs of germination. At the end of the experiment all seeds were dead.

In pure oxygen germination was, in most of the cases, accelerated as compared with other gas-mixtures, but soon the seeds turned brown and became slimy. July 14 normal seeds showed an average root-length of 3.1 cm (1—6) while the heated seeds showed an average root-length of 1.0 cm (0.3—3). The normal seeds already showed leaves and a vegetation-point. Only a few of the heated seeds showed analogous development. Experiments with 100 % O_2 were discontinued in August 1939.

Series IV, atmospheric air, showed no reliable results due to repeated infections.

Series III 85 % N_2, 15 % O_2 and

Series V 40 % N_2, 60 % O_2 were the only ones that showed reliable results.

In 85 % nitrogen and 15 % oxygen both normal- and heated seeds germinated well and rapidly.

Average root-length of normals on July 14 2.6 cm (0.5—6).

,, ,, ,, ,, heated ,, ,, 14 1.3 cm (0.2—3).

50 % of the normal plants showed an average of 3 leaves on August 24.

30 % ,, ,, sterile ,, ,, ,, ,, ,, 3 ,, ,, ,, 24.

On August 24 there were 30 % of cripples in the "normal" series. This number increased to 88 % on November 4 to remain almost constant in atmospheric air (85 % on February 19). Of the heated seeds developed into cripples; 55 % on August 24, 70 % on November 4, while after a sojourn in atmospheric air, this number increased to 85 % on February 19 1940.

In 40 % nitrogen and 60 % oxygen both normal and heated seeds germinated well and rapidly.

Average root-length of normals on July 14 3.5 cm (2 —7).

,, ,, ,, ,, heated ,, ,, 14 2.5 cm (0.2—7).

75 % of the normals and 65 % of the sterile plants showed 3 leaves at this date.

On August 24 there were 55 % cripples in the "normal" series, this number increased to 80 % on February 19 1940.

All of the sterilized seeds yielded cripples on August 24 and showed February 19 1940 swollen roots, a phenomenon not mentioned by DE JONGH.

A number of "normals" which still showed a sharp vegetation-point on August 24, had developed a blunted stem-apex on February 19 1940. This seems the more remarkable as the plants remained in atmospheric air from November until February.

DE JONGH has shown that the number of "spontaneous cripples", developing from normal seeds is extremely variable, while 48 hours heating of the seeds to 40° C. always yielded cripples. In our experiments the lowered oxygen tension *actually caused the development of 45 % normal plants from heated seeds after less than 3 months.* A number of these normals reverted to cripples in atmospheric air.

Also the number of spontaneous cripples in 85 % N_2 + 15 % O_2 (30 %) was lower than that in 60 % O_2 + 40 % N_2 (55 %). A sojourn in atmospheric air for 3 months increased this number in both cases to about 80 %.

In 40 % N_2 + 60 % O_2 the heated seeds only yielded crippled plants after 3 months.

These preliminary results seem to give a strong indication that the hypothesis, stated in the work of PH. DE JONGH, *according to which the bacterial film on the stem-vegetation-point of Ardisia crispa lowers the oxygen tension in the stem meristem, contains elements of truth.*

LITERATURE.

DE JONGH, PH. (1938), On the symbiosis of Ardisia crispa (Thunb.) A. DC. Thesis, Leyden.

WILSON, P. W. and C. E. GEORGI (1932), Bot. Gaz. **94**, 346.

Geology. — *Three geological sections across South Sumatra.* By J. WESTERVELD. (Communicated by Prof. H. A. BROUWER.)

(Communicated at the meeting of October 25, 1941.)

INTRODUCTION.

This paper is ment to give a short summary of the geology of South Sumatra as it has become known by the work achieved by the Sumatra party of the Geol. Survey of the Netherlands East Indies between 1927 and 1933, which methodical survey, the results of which have been published in the Explanations to the Geol. Map of Sumatra, scale 1 : 200.000 [1]), in a certain way may be considered as a continuation of the explorations made previously by A. TOBLER [2]) and others in Palembang and Djambi. The general geological structure of the region concerned, as it can now be conceived from the results of these explorations and besides from the mapping done by employees of oil companies, is illustrated by three sections across the entire width of the island and by two complementary cross-cuts. The writer wishes to thank Dr. H. M. E. SCHÜRMANN and Dr. J. K. VAN GELDER, both at the Hague, and Dr. J. ZWIERZYCKI, now at Warsav (Poland), for many valuable data concerning the oil region put at his disposal.

The area treated in this article covers the Residencies Palembang, Lampoeng Districts and parts of Djambi and Benkoelen [3]). Apart from the presumably pre-Carboniferous and possibly even pre-Palaeozoic crystalline schists and granitic gneisses of the Lampoeng Districts, which primarily belong to an old mountain system of unknown extension and still obscure connections with metamorphic rocks elsewhere in the East Indian Archipelago, the rock formations of South Sumatra outside the Neogene basin of Djambi, Palembang and the Lampoeng Districts are distributed over three main orogenetic units of major importance in the larger western half or "Asiatic" part of the East Indian Archipelago. The sediments of the oil basin were folded in subsidiary and later phases of movements, which culminated in late-Miocene time in the Barisan zone. From NE. to SW. the three main orogenetic stems may be distinguished as: 1. The late-Jurassic *Malaya orogene,* the chief exposures of which are found on the Malayan Peninsula, the Tin Islands (Riouw Archipelago, Banka, Billiton) and in West and Central Borneo (Kapoeas region, Schwaner Mts., upper Mahakam region), whereas on Sumatra it is only visible in the Bt. Batoe SE. of Palembang, the Bt. Iboel in E. Djambi and possibly also in the Tigapoeloeh Mts.; 2. The middle- or late-Cretaceous (or ev. Palaeocene) *Sumatra orogene* [4]), exposed in the

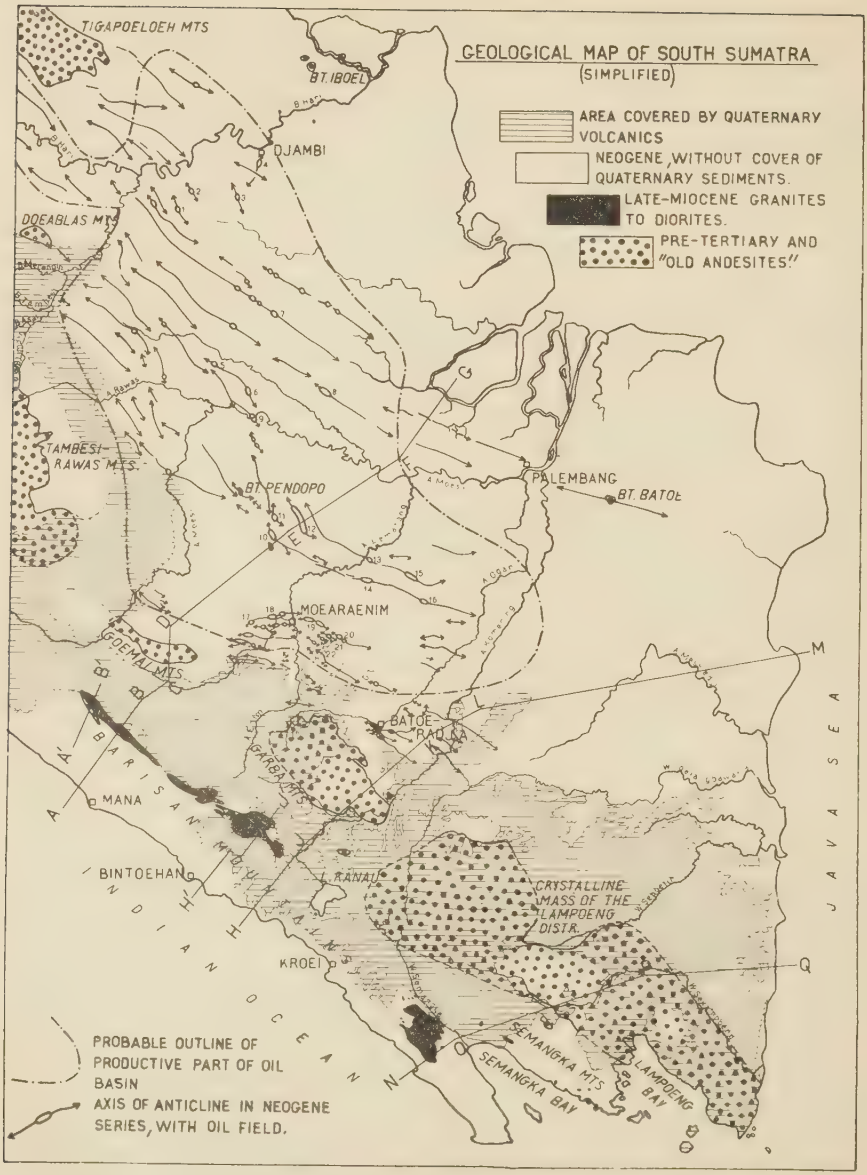
---

[1]) R. W. VAN BEMMELEN, Explanation (Toelichting) to sheet 10 (Batoeradja) and sheet 6 (Kroeï) of the Geological Map of Sumatra (1932, 1933); K. A. F. R. MUSPER, Do. to sheets 15 (Praboemoelih) and 16 (Lahat) (1933, 1937); C. H. VAN RAALTEN, Do. to sheet 7 (Bintoehan) (1937); J. VAN TUYN, Do. to sheet 4 (Soekadana), sheet 8 (Menggala), sheet 13 (Wiralaga) and sheet 9 (Gedongratoe) (1932, 1934, 1934, 1937); J. WESTERVELD, Do. to sheet 5 (Kotaboemi) and sheet 3 (Bengkoenat) (1931, 1933); J. ZWIERZYCKI, Do. to sheet 1 (Teloekbetoeng) and sheet 2 (Kotaägoeng) (1931, 1932).

[2]) A. TOBLER, Topografische und Geologische Beschreibung der Petroleumgebiete bei Moeara Enim (Süd Sumatra). Tijdschr. v. h. Kon. Ned. Aardr. Gen., 2nd series, **23**, 199—315 (1906); Do., Djambi-verslag. Jaarb. v. h. Mijnw. in Ned. O.-Indië, **48**, Verh. III, pp. 585 (1919). With Atlas.

[3]) "oe" of Netherlands orthography to be pronounced as "u".

[4]) In a previous paper (J. WESTERVELD, „De tektonische bouw van Zuid-Sumatra", Handel. v. h. 28ste Nederl. Nat.- en Geneesk. Congres, p. 264—267, 1941), the writer has



Scale 1 : 3.347.140.

Main oil fields:

1. Betoeng; 2. Badjoebang; 3. Tempino; 4. Kenali Assam; 5. Soeban Boeroeng; 6. Babat; 7. Sumpal; 8. Kloeang; 9. Mangoendjaja; 10. Benakat; 11. Djirak; 12. Talangakar-Pendopo; 13. Goenoengkemala; 14. Limau; 15. Talangdjimar; 16. Tandjoengmiring; 17. Soengei Ramok; 18. Tandjoeng Loentar; 19. Minjak Itam-Kampoeng Minjak; 20. Soeban Djerigi; 21. Batoekras; 22. Soengei Taham.

Tambesi-Rawas Mts., Goemai Mts., Garba Mts. and in the Crystalline Mass of the Lampoeng Districts, all cupolas forming part of a broad mountain system, which can be followed without interruption from the Tambesi-Rawas Mts. onward along the whole length of Sumatra between NW. Palembang and the north-point of Atjeh; 3. The late-Miocene *Soenda orogene*, represented by the folded Miocene sediments, effusive rocks and irruptives of the Barisan and Semangka Mts., which zone can be followed from S. Benkoelen until the SW.-part of the Government West Coast of Sumatra, where it evidently plunges below sea-level opposite the coast region of Tapanoeli, where the rocks of the Sumatra orogene are washed at various places by the Indian Ocean.

The Sumatra orogene forms the backbone of the island in Central and North Sumatra. On Java this late-Mesozoic mountain system is only exposed in the small pre-Tertiary cores of the South Serajoe Mts. and Djiwo Hills, while on the east-side of the area covered by the Malaya orogene equivalent structures of about the same age are very probably developed in the Meratoes and Bobaris Mts. of SE. Borneo.

The Soenda orogene, characterized by the occurrence of Miocene deposits in dominantly volcanic facies and by the development of Neogene, dacitic and dioritic to granitic intrusiva along its entire area of extension, can be traced from the Barisan Mts. in Benkoelen towards the southern ranges of Java, whence it takes its further course over the Lesser Soenda Islands as far as the Banda sea, from which region it proceeds with a sharp bend into the SW.-arm of Celebes and farther on into the western part of Central Celebes and the north-arm of this island. G. L. SMIT SIBINGA [5]) proposed the denomination "Soenda orogene" for the total assemblage of mountain structures developed on Sumatra, Java, the Lesser Soenda Islands and in the West Arc of Celebes. The name will be used here in a more restricted sense, applying it only to the Miocene magmatic zone indicated above.

STRATIGRAPHY AND IGNEOUS ROCKS.

Crystalline schists and gneisses, which presumably must be of at least pre-Carboniferous age in view of the entirely non-metamorphic state of the Anthracolitic formations of Sumatra, constitute the frame-work of the Crystalline Mass of the Lampoeng districts (section N—Q), while smaller exposures are found between Lake Ranau and the Garba Mts. and between the old-Miocene rocks of the Barisan Mts. (districts Mana and Bintoehan, sections A'—B', H'—J'). This crystalline basement of S. Sumatra, which is partly covered by Miocene deposits and by products of Quaternary and pre-Neogene volcanism, can be traced from Lampoeng Bay in a NW.-direction until the Mana section of the Barisan Mts. It represents the only definite example of crystalline schists and gneisses on Sumatra and consists of an assemblage of orthogneisses (metamorphic granites, quartz-diorites, quartz-porphyrites and tuffs), biotite-, muscovite- and chlorite-schists, sericite-quartzites, graphitic zoisite-garnet-mica-schists, massive and schistose amphibolites, albite-sericite-schists and occasional marble beds; all rocks with a degree of crystallinity corresponding to the epi- to meso-zone of regional metamorphism as distinguished by U. GRUBENMANN.

Permocarboniferous rocks. A *Permian* rock suite of porphyrites and dark limestones with

---

suggested a late-Cretaceous or ev. old-Eocene age for the Sumatra orogene and its post-orogenetic irruptive rocks. The lower age limit, however, has to be removed to the middle-Cretaceous, as all Cretaceous deposits of S. Sumatra presumably belong to the lower division of this formation.

The Malayan prefixes in this paper have the following meaning: Bt. = Boekit = Mountain top or hill; A. = Aer = River or water; B. = Batang = River; W. = Wai = River or water; S. = Soengei = River.

[5]) G. L. SMIT SIBINGA, The Malay double (triple) orogen. Proc. Kon. Akad. v Wetensch., Amsterdam, **36**, 202—210, 323—330, 447—453 (1933).

Glomospira, Neoschwagerina and *Fusulina* sp. is known from the pre-Neogene core of the Bt. Pendopo on the Talangakar anticlinorium in Central Palembang [6]). Late-Palaeozoic sandstones, conglomerates and liparitic to andesitic effusive rocks, together with limestones, which yielded a fauna of gastropods, brachiopods and Foraminifera of probably *lower-Permian* age, are developed at the Tambesi river on the NE.-side of the Tambesi-Rawas Mts. [7]). NW. of this locality *upper-Carboniferous* (to lower-Permian?) rocks in dominantly volcanic facies are developed in a broad syncline in Upper Djambi [8]). Pebbles of Permocarboniferous Fusulina limestone were besides found in a conglomerate bed between Middle Palembang beds NW. of Palembang town [9]), so that late-Palaeozoic rocks must form part of the pre-Neogene basement formations along the Palembang anticlinorium.

Trias (?). An indication to the existence of Triassic sediments in the Mesozoic slate formations of the Tambesi-Rawas Mts. is afforded by a find of *Thecosmilia* sp. in limestones along the Tambesi river [10]). Triassic sediments are probably to be expected in the Tigapoeloeh and Doeablas Mts., the sedimentary rocks of which consist largely of strongly folded slates, interbedded with sandy layers, calcareous phyllites, a few marble beds and occasional volcanic material [11]). Triassic are possibly also the quartz-sandstones exposed on Bt. Iboel.

Jurassic sediments are exclusively known with certainty from the Tambesi-Rawas Mts., where a small Dogger pelecypod fauna was found in sandy shales along the Temalang river N. of the Rawas river [12]). *The S. Temalang shales are the least disputable Jurassic beds thus far discovered on Sumatra.*

Cretaceous deposits are known from the NE.-margin of the Tambesi-Rawas Mts., the Goemai Mts., the Semangka Mts., and very probably also from the Garba Mts. In the Tambesi-Rawas Mts. dull shales and limestones on the B. Asai and B. Limoen yielded a Valanginian fauna of Ammonoidea (e.g. *Neocomites* and *Thurmannites* sp.), pelecypods, gastropods and echinids [13]). In the Goemai Mts. (section A—G), the pre-Tertiary rocks of which have all proved to be Cretaceous [14]), the late-Mesozoic deposits were divided into two apparently conformable series, distinguished resp. as the Saling beds or upper division (thickness at least 400 m.) and the Lingsing beds or lower division (thickness at least 3000 m.). The Saling beds are composed of volcanic breccias, tuffs and basaltic to andesitic lavas with a greenstone appearance, interbedded with fossiliferous limestone banks, which yielded a probably lower-Cretaceous fauna of Nerineas, Lovceniporas, Lof-

---

[6]) Jaarb. v. h. Mijnw. in Nederl.-Indië, **60**, Alg. Ged., 25 (1931).

[7]) A. TOBLER, Op. cit., 195—200, 293, 294 (1919).

O. E. MEYER, Brachiopoden des Perm und Unter-Carbon der Residentschaft Djambi (Sumatra). Verh. v. h. Geol. Mijnb. Gen. voor Nederl. en Kol.; Geol. serie, **5**, 203—222 (1920—1922).

[8]) J. ZWIERZYCKI, Die geologischen Ergebnisse der Paläobotanischen Djambi-Expedition 1925. Jaarb. v. h. Mijnw. in Nederl.-Indië, **59**, Verh. II, 1—70 (1930).

[9]) Netherlands East Indies Mining Service; quarterly report (1932).

[10]) A. TOBLER, Op. cit., 78 (1919).

[11]) A. TOBLER, Op. cit., 206—212, 310—315, 416 (1919).

[12]) F. FRECH and O. E. MEYER, Mitteljurassische Bivalven von Sungi Temalang im Schieferbarissan (Residentschaft Djambi). Verh. v. h. Geol. Mijnb. Gen. voor Ned. en Kol.; Geol. serie, **5**, 223—229 (1920—1922).

[13]) E. BAUMBERGER, Die Kreidefossilien von Dusun Pobungo, Batu Kapur Menkadai und Sungi Pobungo (Djambi, Sumatra). Verh. v. h. Geol. Mijnb. Gen. voor Ned. en Kol.; Geol. serie, **8**, 17—47 (1925).

[14]) K. A. F. R. MUSPER, Nieuwe fossielresten en de ouderdom der kalksteenen in het Pretertiair van het Goemaigebergte. De Ing. in Ned.-Indië; IV. Mijnbouw en Geologie, **1**, 521—531 (1934).

Do., Explanation to sheet 16 (Lahat) (1937).

tusias, echinids, small Foraminifera and corals. The Lingsing beds are a series of thin-bedded shales, tuffaceous marls with Radiolaria and sponge spicules, Radiolaria-bearing hornstones, and an occasional limestone bed with small Foraminifera and *Orbitolina* sp. of a presumably lower-Cretaceous type. In the Semangka Mts. a strongly folded series of black shales, tuffaceous sandstones, beds of coral limestone and of black limestone with *Orbitolina* sp., also of a presumably lower-Cretaceous type, was found abutting on the crystalline rocks of the Lampoeng Mass (section N—Q). The Cretaceous age of the Upper and Lower Garba beds (section H—M), into which the pre-Tertiary deposits of the Garba Mts. were divided, is suggested by their great lithological similarity to the Cretaceous rocks of the Goemai Mts. The Upper Garba beds, only preserved in a few steep synclines, consist of tuffaceous marls with Radiolaria and of diabase tuffs. Similar beds are also exposed in the cores of some anticlines near Batoeradja. The Lower Garba beds are a strongly folded series of hydrothermally altered, andesitic, basaltic and dacitic tuffs and lava flows, interbedded with shales and limestone beds.

Although definite evidence is still lacking, *a transgressive position of the lower-Cretaceous* is suggested by the side-by-side occurrence of Cretaceous rocks and crystalline schists in Palembang and the Lampoeng Districts, apparently without any intercalation of Permocarboniferous and old-Mesozoic formations, which play such an important part in Central and North Sumatra.

Pre-Tertiary intrusive rocks, granites to diorites and allied types, are known from all the pre-Tertiary ranges. In the Tambesi-Rawas Mts., Goemai Mts., Garba Mts., the Crystalline Mass of the Lampoeng Districts, and possibly also in the Doeablas Mts. and the pre-Tertiary core of Bt. Pendopo, they probably all belong to one post-orogenetic family, intimately connected with post-lower-Cretaceous folding in the zone of the Sumatra orogene. Their period of intrusion has to be located between the lower-Cretaceous and the Neogene, but a pre-Eocene age seems most probable in view of the fact that in Atjeh (N. Sumatra) the already deeply eroded Sumatra orogene is locally covered unconformably by Oligocene strata. The granites of the Tigapoeloeh Mts. and the granosyenite of Bt. Batoe probably belong to the late-Jurassic Malaya orogene; almost certainly the rock of Bt. Batoe, which by its high content of potash and rare earths [15]) shows unmistakable relationship to the eruptive rocks of the Tin Islands. Still older epochs of magmatic activity are represented by the undatable granite-gneisses of the Lampoeng Districts and by detached bodies of granitic rocks, which occur together with the Permian deposits of the B. Tambesi in an overthrust mass above Mesozoic slates [16]). A homologue of this overthrust granite is found in the B. Merangin-B. Mesoemai region in Upper Djambi, where a pre-upper-Carboniferous granite of analoguous tectonic position occurs at the base of the volcanic upper-Carboniferous series described by J. ZWIERZYCKI [17]).

The plutonic bodies of the Tambesi-Rawas Mts. are intrusive into Mesozoic slates and Jurassic (to Cretaceous?) limestones (pyrometasomatic iron and Cu-Pb-Zn ores!), while those of the Goemai and Garba Mts. are resp. intrusive into the Lingsing-Saling beds and the Garba beds (sections A—G, H—M). The Crystalline Mass of the Lampoeng Districts contains many intrusive granites (pyrometasomatic iron ores!), the most extensive body of which is exposed between the upper courses of W. Sekampoeng and W. Sepoetih (section N—Q). Dominant rock types in all these cupolas are granite to diorite, with on a minor scale gabbro and dikes of aplite, pegmatite, pyroxenite, porphyrite, dolerite, spessartite and kersantite. In the Doeablas Mts. the pre-Tertiary slate formation is in-

---

[15]) W. VAN TONGEREN, Mineralogical and chemical composition of the syenite-granite from Boekit Batoe near Palembang, Sumatra, Netherlands East Indies. Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 670—673 (1936).

Do., On the occurrence of rare elements in the Netherlands East Indies, etc. Dissertation, Utrecht; analysis, p. 148 (1938).

[16]) A. TOBLER, Op. cit., 195—200, 293, 294 (1919).

[17]) J. ZWIERZYCKI, Op. cit. (1930).

truded by granite, augite-syenite, essexite and dikes of diorite-porphyrite, hornblende-syenite, porphyry, aplite, tourmaline-bearing pegmatite and spessartite, while the plutonics of the Tigapoeloeh Mts. consist of granite to diorite, joined by veins of aplite and pegmatite [18]).

"Old andesites" [19]). The general transgression of the lower-Miocene sea over the eroded structures of the Sumatra orogene was preceded in Eocene or early-Miocene time by a stage of terrestrial, dominantly andesitic volcanism, the frequently propylitized products of which still cover wide areas over the NW.-section of the crystalline Lampoeng Mass, the NW.-part of the Garba Mts. and the S.-part of the Tambesi-Rawas Mts. The Lower Kikim tuffs around the Goemai Mts. and andesitic rocks which occur between the pre-Tertiary rocks of the Bt. Pendopo (Permian and intrusive granite) and their Miocene cover also belong to these effusives. Part of the Tertiary andesites, breccias and tuffs of the Barisan and Semangka Mts. may probably be correlated with these "old andesites", but it is hardly possible to distinguish them from the old-Miocene volcanics of these mountain ranges. The "old andesites" are sometimes cut by Au-Ag veins of an epithermal type (NW.-section of the Lampoeng Mass, Tambesi-Rawas Mts. S. of the A. Rawas, in the latter region apparently associated with dacites and liparites).

Neogene sediments and volcanics. The following stratigraphical scheme applies in a general way to the Neogene formations of S. Sumatra (oil basin and Barisan zone):

Oil basin in SW. Palembang.			Barisan and Semangka Mts.
Upper, Middle, Lower	Palembang beds (younger-Miocene and Pliocene).		Marine Pliocene of the Benkoelen coast and Semangka valley.
			~~ unconformity.
			"Intermediary Neogene" of Benkoelen (Lower Palembang beds?).
			~~ local unconformity or stratigraphical gap?
Batoeradja stage	Telisa or Goemai beds.	Batoeradja-Telisa series (older Miocene)	Batoeradja — Telisa series (at many places chiefly in volcanic facies).
	Batoeradja limestone.		
	Basal beds (quartz-sandstone horizon(s), shales, layers of black lignite or wood horizons, sometimes volcanic deposits).		

The Neogene deposits of the oil basin attain their greatest thickness in the region E. of the Goemai Mts., where the Telisa and Palembang beds alone together measure about 7600 m. (Telisa beds ± 4800 m.; Lower Palembang beds 700—1000 m.; Middle Palembang beds 650—800 m.; Upper Palembang beds 500—1000 m.), which figure ought to be augmented with that for the Batoeradja stage, which around the Goemai Mts. consists of a basal quartz-sandstone horizon (75—200 m.), covered by a volcanic series of andesitic breccias, tuffs and lavas (the Upper Kikim tuffs) with an aggregate thickness of 2000—2500 m. on the S.-side of the range and 300—950 m. on its N.-flank. Along the northern margin of the Goemai Mts. the Telisa and Palembang beds show a very reduced thickness and the same phenomenon can be observed in the area of the Batoeradja folds and in the Lampoeng Districts, where the Neogene series covers a relatively stable region of slow former subsidence. In the case of the Goemai anticline its rather thin cover of Neogene deposits, especially on the N.-side, can be explained as a consequence

[18]) A. TOBLER, Op. cit., 379—381, 388, 389 (1919).

[19]) The term has also frequently been used to denote all Tertiary andesites.

of its early rising in a initial stage of the process of young-Tertiary folding. The Neogene stratigraphy of the Moearaenim region also applies more or less to Central and North Palembang and Djambi. Along the E.-side of the Tambesi-Rawas Mts., however, the entire Neogene series shows a little differentiated, tuffaceous development, while E. and SE. of the Garba Mts. the Neogene beds are developed in sandy, tuffaceous and conglomeratic facies. The Telisa series wedges out between its pre-Neogene basement and the Palembang beds along the Palembang anticlinorium (section A—G), so that the Lower Palembang beds in this region are lying immediately above pre-Tertiary rocks.

The *Batoeradja stage* is especially well-developed around the Goemai and Garba Mts. and in the anticlines near Batoeradja. Its basal beds are of less importance in the latter region, where the old-Miocene Batoeradja limestone (Tertiary e4—e5 according to W. LEOPOLD and I. M. VAN DER VLERK's scheme for the East Indian Tertiary [20]), characterized by Eulepidinas, Spiroclypei, *Miogypsina dehaarti* V. D. VLERK and *Trillina Howchini* SCHLUMB. as guide fossils, acquires a maximal thickness (up to 300 m.). In small basins along the upper courses of W. Sekampoeng and W. Sepoetih on either side of the Lampoeng Mass the basal beds consist of some hundreds of meters of quartz-sandstones and quartz-conglomerates with a few coal beds and capped by Batoeradja limestone. SW. of the Crystalline Mass this basal Miocene series passes side-ways into the volcanic old-Miocene of the Semangka Mts. (section N—Q). The Batoeradja stage merely represents a facial type of the lowest division of the Telisa series. On the Talangakar anticlinorium, where the Telisa beds have a thickness of 1000—1200 m., it is developed as a sandy deposit and contains an important oil horizon (oil fields Talangakar-Pendopo, Benakat etc.). In the Barisan and Semangka Mts. the old-Miocene, the Telisa beds inclusive, is largely developed as a series of andesitic lavas, tuffs and breccias with some sedimentary intercalatations, e.g. shales and limestones with Batoeradja guide fossils. Only near Mana the Telisa beds show their normal sedimentary development.

The Telisa beds (upper part of Tertiary e and T. f) are a monotonous series of Globigerina marls and shales with intercalations of andesitic tuffs and breccias, beds of glauconitic sandstone, platy or concretionnary limestones and occasional layers with plant remains. Spyroclypei were still found in the lower part of the series in association with Lepidocyclinas and other genera. In the Barisan Mts. near Mana the Telisa beds are developed in normal sedimentary facies (thickness 3000—4000 m.), but in the districts Bintoehan and Kroei and in the Semangka Mts. the series is almost entirely replaced by andesitic volcanics (see sections). The Telisa beds contain important oil horizons in Central Palembang and Djambi.

The *Lower Palembang beds* (late-Miocene) are a series of bluish green or gray, frequently glauconitic marls and claystones, interbedded with tuffaceous sandstones and glauconitic, concretionnary, marly limestones. The formation contains abundant molluscs and Foraminifera (Lepidocyclynas, Miogypsinas, Cycloclypei, rotalides and other genera) and corresponds more or less to the upper-Miocene Badoei, Bodjongmanik, Njalindoeng and Tjilanang beds in W. Java [21]). The L. P. beds contain oil sands practically over the whole extension of the oil basin in Palembang and Djambi.

The *"intermediary Neogene"* of the Benkoelen coast region (distr. Mana and Bintoehan) presumably represents a stratigraphical equivalent of the L. P. beds, to which they correspond facially. NE. of Mana this series (thickness ± 1500 m.) apparently succeeds in

---

[20]) W. LEUPOLD and I. M. VAN DER VLERK, De Palaeontologie en Stratigrafie van Nederlandsch Oost-Indië; "The Tertiary". Leidsche Geol. Meded., **5**, 611—648 (1931).

[21]) Guide fossils of the Bodjongmanik and Badoei beds in Central Bantam, W. Java (C. H. OOSTINGH, Mollusken als gidsfossielen voor het Neogeen in Nederlandsch-Indië. Hand. v. h. 8ste Ned.-Ind. Natuurw. congres, 508—516 (1938)), also found in the L. P. beds are: *Turritella subulata* MART., *Amusium hulshofi* MART., *Apolymetis grimesi elongata* (HAANSTRA and SPIKER), *Cultellus dilatatus* MART., *Siphocypraea caput-viperae* (MART.).

conformable succession to the Telisa beds, although the mutual relations are obscured by a reverse fault (sect. A'—B', A—G). Near Bintoehan this series, which is generally rich in shell fragments, contains also pumice tuffs, silicified wood and silicified dacite tuffs, while it is probably separated from the Telisa beds in this district by a stratigraphical gap. The marine fauna of these beds mainly consists of molluscs and Foraminifera (Lepidocyclinas, Miogypsinas, Cycloclypei, e.g. *Cycloclypeus* ex. gr. *guembelianus* BRADY and *C. (Katacycloclypeus) annulatus* MART., which two species according to TAN SIN HOK [22]) are resp. restricted to the upper-Miocene to Pliocene and to the upper-Miocene).

The *Middle Palembang beds* near Moearaenim (late-Miocene to lower-Pliocene) consist of claystones, tuff-sandstones, interbedded with marly or glauconitic concretions, glauconite sands and several groups of brown-coal beds (total thickness of coal beds up to 47—90 m., but strongly variable). Marine horizons with molluscs, echinoderms, shark teeth and small Foraminifera are found in the lower part of this formation near the Goemai Mts. Oil horizons are known mainly on the Moearaenim anticlinorium.

The *Upper Palembang* beds (Pliocene to Pleistocene?) consist principally of acid pumice tuffs, tuff sands and kaoline-like clays, practically without marine horizons and with few coal strings. The formation contains no oil.

The *Pliocene of Benkoelen* and the Semangka valley lies unconformably above all older Neogene formations and shows a monoclinal, sea-ward dip of 5—10°, only locally interrupted by faults, in the coast region of Benkoelen. The formation is of marine-tuffaceous origin, contains abundant mollusc remains and consists of greenish, tuffaceous marls, calcareous marls, tuffaceous claystones, layers of pumice tuff, sometimes massive dacitic-liparitic tuffs, andesitic ocnglomerates and occasionally also andesitic lavas (total thickness of the series 800—1000 m.). Plant remains and layers of impure brown-coal are also frequently found. The series corresponds more or less to the upper-Pliocene Tjimantjeuri, Tjilegong, Bodjong, Kaliglagah and Sondé beds in W. and Central Java [23]).

During the Batoeradja-Telisa epoch the Neogene sea of the oil basin stood in close connection with the Indian Ocean through wide passages around the Goemai and Garba Mts., which condition still persisted, albeit perhaps on a reduced scale, during the Lower Palembang and even locally — NW. of the Goemai Mts. — during the Middle Palembang epoch. Before the Upper Palembang epoch, however, all connections were apparently cut off by the rise of the Barisan zone and the Lampoeng Mass; the marine Pliocene of the west coast possibly corresponds to the terrestrial-volcanic U. P. beds on the E.-side of these structural units.

Late-Miocene intrusive rocks. The old-Miocene sediments and volcanics of the Barisan and Semangka Mts. are penetrated at many places by granitic to dioritic (on a minor scale also gabbroic) and dacitic rocks in bosses or batholithic bodies. The largest of these intrusions are found in the southern peninsula of Benkoelen, NE. of Bintoehan and NE. of Kroei, while similar intrusions are also found in the coast range of N. Benkoelen and in the region SE. of Padang (Gov. West Coast of Sumatra). The late-Miocene granites to diorites are responsible for extensive propylitizations and silicifications of the old-Miocene volcanics in the Barisan zone, while they have also to be considered as the parent rocks of the epithermal Au-Ag veins in N. Benkoelen (Lebong region) and near Padang. The massive dacites are intimately associated with the holocrystalline rocks, but in most cases seem to be slightly older.

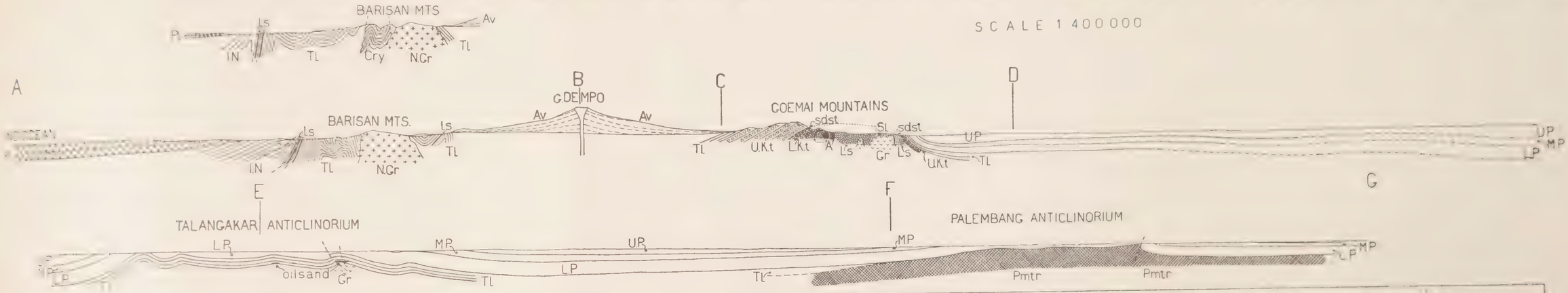
---

[22]) TAN SIN HOK, On the genus Cycloclypeus Carpenter. Dienst v. d. Mijnb. in Ned.-Indië; Wetensch. Meded., **19** (1932).

[23]) C. H. OOSTINGH, Op. cit. (1938); also: Note on the stratigraphical relations between some Pliocene deposits in Java. De Ing. in Ned.-Indië; IV. Mijnb. en Geol., **6**, 140—141 (1939). Guide fossils of the Tjilegong and Bodjong beds (Bantam, W. Java) also found in the Benkoelen Pliocene are: *Terebra insulindae* P. J. FISCHER, *Nuculana praeradiata* (O. BOETTG.), *Turritella angulata bantamensis* MART., *Polinices (Neverita) sulcifer* (MART.), *Pecten javanus* MART., *Cardita javana* MART.

GEOLOGICAL SECTIONS ACROSS SOUTH SUMATRA

SCALE 1 400 000



Cry=Crystalline schists and gneisses — Pmtr=Basement rocks of the Soenda Land (Permocarboniferous, Trias and intrusive granites)

Ls=Lingsing series — Sl=Saling series — Cretaceous

L.Cb=Lower Carba series — U.Cb=Upper Carba series — Cretaceous?

Cr=Cretaceous, undifferentiated

sdst.=Quartz-sandstone — Bls=Batoeradja limestone — U.Kt=Upper Kikim tuffs — Tl=Telisa beds — ls=Foraminifera limestone in Telisa beds

Batoeradja-Telisa series (older Miocene)

Gr=Pre-Neogene granite (middle- or upper-Cretaceous)

N.Gr=Neogene (late-Miocene) granite

Da=Late-Miocene dacite — A=Intrusive andesite

L.P=Lower Palembang beds — M.P=Middle Palembang beds — U.P=Upper Palembang beds — Younger Miocene-Pliocene

Pl=Marine Pliocene of the West Coast

N=Neogene, undifferentiated

I.N=Intermediary Neogene of the West Coast (Lower Palembang beds?)

Al=Marine Quaternary and alluvial deposits

L.Kt=Lower Kikim tuffs (of pre-Batoeradja age) — O.A=Andesitic lavas, breccias and tuffs (of Batoeradja-Telisa age and older)

Av=Andesitic volcanoes and their products — Dt=Dacitic tuffs — Rt=Ranau tuffs — Lp=Liparite — Bz=Basaltic lavas

Quaternary volcanism

Quaternary sediments and volcanics. Marine Quaternary sediments are represented by upheaved coral reefs along the Benkoelen coast, especially near Bintoehan (up to 50 m. above sea-level); furthermore by extensive blankets of clay deposits with marine shells, which form large embayments between the Neogene hills of E. Palembang and E. Djambi. The marine Holocene beds on the eastern outskirts of the oil basin were formed during the post-Pleistocene period of partial submergence of the eastern coast region. Only in very recent time the sea again receded over a height of 5—10 m.[24]). *Quaternary stages of volcanism* produced the following five groups of effusive rocks, tuff mantles and volcanic cones: 1°. *Liparitic-perlitic lavas of fissure eruptions in the Pasoemah region* (SW. of the Goemai Mts.) *and along the Semangka valley*; 2°. *Dacitic pumice tuffs and dacitic-liparitic necks of the Lampoeng Districts*; 3°. *Andesitic to basaltic volcanoes* within and NE. of the Barisan zone; 4°. *Liparitic-dacitic pumice tuffs of the Ranau eruption,* deposited around Lake Ranau and E. of the Garba Mts.; 5°. *Basaltic lavas of the Soekadana fissure eruptions* in the eastern Lampoeng Districts. Groups 1 and 2 are the oldest and connected with longitudinal fault zones as channel ways. The acid necks of the Lampoeng Districts NW. of Lampoeng Bay (two of them indicated on section N—Q) are probably the eruption points of the acid Lampoeng tuffs. The Ranau tuffs, connected with the formation of the volcano-tectonic L. Ranau depression, are older than the youngest andesitic volcanoes, but younger than the oldest andesitic cones. The andesitic volcanoes are mainly located on diagonal or transverse fault zones.

TECTONICS.

The rocks of the Sumatra orogene, which were folded in the middle- or late Cretaceous (or ev. Palaeocene) epoch, are strongly, but not excessively folded; dynamo-metamorphism or regional metamorphism are apparently not developed on an intensive scale. Overthrusts of presumably not very large horizontal displacement are known from the NE.-flank of the Tambesi-Rawas Mts., where Permian rocks and detached masses of granite were thrown in a SW.-direction upon Mesozoic slates; furthermore from the SW.-flank of the Lampoeng Mass W. of Lampoeng Bay, where crystalline schists are separated by a NE.-ward dipping, reverse fault from the Cretaceous rocks of the Semangka Mts. (section N—Q). The Neogene deposits were strongly folded and also cross-faulted in the Barisan section of the larger Soenda orogene in late-Miocene time before the intrusion of the late-Miocene granites (to diorites) and dacites. Later movements mainly affected the Mio-Pliocene deposits of the oil basin, which were warped into three main anticlinoria in South and Central Palembang: the Moearaenim-Batoeradja anticlinorium, the Talangakar anticlinorium and the Palembang anticlinorium. In N. Palembang the Talangakar and Palembang anticlinorium unite into one broad system of folds. In the oil region the most complicated structures are found in the oil fields of the Moearaenim anticlinorium, which are segmented by cross faults (Soeban Djerigi oil field) or even struck by overthrusts (Batoekras and Minjak Itam-Kp. Minjak oil fields). Reverse faults (with NE.-ward dip), however, are also known from the folds on the Talangakar anticlinorium, as for instance on the Talangakar-Pendopo oil field (secion A—G), along the SW.-flank of Bt. Pendopo, along the SW-side of the Karangringin oil field (S. of the Mangoendjaja oil field) and on various folds between the A. Moesi and A. Rawas. A steep flexure or even reverse fault is besides found on the NE.-flank of the Palembang anticline (secion A—G). The arcual shape of the Moearaenim fold system may be explained as the result of the strong NE.-ward push exerted upon the mighty succession of poorly consolidated, Neogene beds in the former area of subsidence NE. of the Goemai Mts. through the tectonic depression between the Goemai and Garba Mts. The Barisan zone only underwent a gradual uplift and dissection into longitudinal fault blocks during the Pliocene and afterwards.

Amsterdam, Geological Institute of the University.

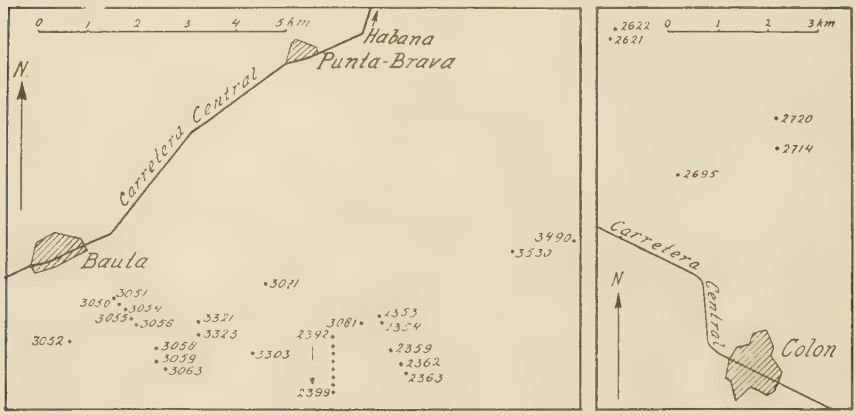
---

[24]) J. VAN TUYN, Over een recente daling van den zeespiegel in Nederlandsch Oost-Indië. Tijdschr. v. h. Kon. Ned. Aardr. Gen., 2nd series, **49**, 89—99 (1932).

Palaeontology. — *Smaller Foraminifera from the Lower Oligocene of Cuba.* By R. C. VAN BELLEN, J. F. C. DE WITT PUYT, A. C. RUTGERS and J. VAN SOEST. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of October 25, 1941.)

Some years ago the geological institute of the university of Utrecht received from the Bataafsche Petroleum Maatschappij at The Hague a collection of washing-residues of marly clays from Cuba. The localities have been indicated on the map; they are situated to the WSW of Habana and in the neighbourhood of the town of Colon. The geologists of the B. P. M. regarded these claymarls as oligocene. The results of the study of this collection are to be found in this communication. In the tabular enumeration at the end of this communication have been indicated: 1. the different species and varieties which compose the faunula, 2. the localities, where each faunal element has been found, 3. the literature for each species, 4. data on the occurrence of each species in the Upper Cretaceous and in the great subdivisions of the Tertiary of Tropical America and in the recent fauna.



There have been found 140 species and varieties, 12 of which are new. The following may be said with regard to the age of the faunula.

8 species have been found in the Upper Cretaceous of Tropical America. Of these, 7 do not have any stratigraphical value, as they have lived as long as the Miocene, or, partly do occur still in the recent fauna. *Spiroplectoides clotho* is the only species which has been found until now exclusively in upper cretaceous deposits; its determination, however, is uncertain. Our faunula, therefore, shows no relation at all with upper cretaceous faunae.

Not less than 61 species and varieties have been found in the Eocene of Tropical America. More than the half — 38 — are known also from the Oligocene of Tropical America, and 7, although not known from the Oligocene, have been found in younger tertiary deposits of Tropical America or occur in the recent fauna. A large percentage of the faunula, 16 species, viz. *Gaudryina cubana, Clavulinoides cubensis, Cl. havanensis, Cl. marielinus, Karreriella chapapotensis, Listerella venezuelana, Planularia westermanni, Nonion mexicanum, Bolivina malkinae, Uvigerina havanensis, Siphogenerina transversa*

var. *cubensis, Valvulineria jacksonensis, Globigerina eocaena, Ruttenia coronaeformis; Planulina cocoaensis* and *Planulina cocoaensis* var. *coöperensis* has been found until now in Tropical America exclusively in the Eocene; 6 of these are restricted to Cuba. It is clear that the fauna has strong eocene affinities. On the other hand, the absence of the genus *Hantkenina*, so common in eocene deposits with this facies suggests a posteocene age.

Still more species and varieties — 75 — are known from the Oligocene of Tropical America. Of these, 38 occur also in the Eocene, and 13, whilst not being known from the Eocene have been found in miocene deposits or occur in the recent fauna. Not less than 24 species seem to be confined in Tropical America to the Oligocene, viz.: *Textularia mississippiensis* var. *alazanensis, Dorothia cylindrica, Cristellaria kemperi, Cr. peregrina* var. *matanzana, Cr. subpapillosa, Planularia thiadensi, Marginulina sublitua, Nodosaria* aff. *carinata, N. matanzana, N. obliquata, Saracenaria schencki, S. senni, Vaginulna elegans* var. *mexicana, Lagena sulcata* var. *strumosa, Gümbelina cubensis, Bolivina matanzana, Uvigerina canariensis* var., *U. cubana, Eponides byramensis* var. *cubensis, Siphonina advena, Anomalina alazanensis, Planulina alazanensis* and *Cibicides mexicanus*; of these, 10 have been found exclusively in Cuba. The affinities with oligocene faunae are apparently very great.

Only 40 species and varieties are known from the Miocene of Tropical America, and 29 of these have been found also in the Oligocene or Eocene of the region. Affinities with the Miocene, therefore, certainly exist, but they are much smaller than the affinities with the Eocene and Oligocene.

If all the washing-residues are from one and the same formation, there is no doubt that the faunula is of oligocene age, and, given the strong relations with the Eocene, may be regarded as Lower Oligocene. It is, however, very well possible that not all the samples are of the same age. The samples 2353—2363 f.i. which are located along a line about vertical to the general strike of the island may very well belong to lower and upper parts of one formation or even to two formations. Generally, the faunula of each sample is, however, too poor to discuss its age separately. Our conclusion is, therefore, that the general aspect of the whole of the samples is so that it suggests a lower oligocene age, that, therefore, certainly a part, probably even the greatest part of the samples must belong to the lower oligocene, but that it is possible that some of the samples belong to an older or younger stratigraphical unit.

A description of the new species and varieties and some remarks on other species are following here.

Gaudryina jacksonensis CUSHMAN (lit. 18, 32).

Our specimens differ from CUSHMAN's description by: 1. too large dimensions (average length 2.5 mm; maximum length 3.6 mm; breadth 1.1—1.75 mm), 2. having hardly visible sutures on triserial part. Moreover, there are in our material transitional forms between *G. jacksonensis* and *G. trinitatensis*. *G. trinitatensis* is triangular in cross section in the triserial part of the test, subcircular in the biserial part, where ribs are lacking. *G. jacksonensis* is triangular throughout, while ribs are lacking only on the last two chambers. The ribs on the transitional forms occur on some of the older biserial chambers, but are lacking on the last 3—5 chambers, which are more inflated than the preceding ones.

Planularia brandsi n. sp. DE WITT PUYT. (Fig. 37).

Test much flattened, spirally coiled, with a broad and thin keel. Chambers very distinct, strongly inflated, regularly increasing in size. Sutures limbate, depressed, straight, somewhat broadening towards periphery. Length 2 mm; breadth 1.3 mm; thickness 0.44 mm; thickness of strongly inflated proloculum 0.68 mm.

Planularia westermanni PIJPERS. (Fig. 34).

For a full discussion of this species see lit. 79. In this publication *Pl. venezuelana* has been placed in the synonymy of *Pl. westermanni*. By a study of Cuban oligocene material,

recently received in Utrecht it turned out that *Pl. venezuelana* must be regarded as a distinct species.

Planularia thiadensi n. sp. DE WITT PUYT. (Fig. 38).

Synon. *Planularia* sp. in lit. 77, p. 94, pl. 16, fig. 2, 3.

Two specimens of our material agree very well with the form which THIADENS described from the oligocene of Santa Clara, Cuba. It differs from *Pl. cubensis* BERM. by its dimensions, by the curved "dorsal" and the straight "ventral" outline, by the broader sutures, the large number and the shape of chambers. It is very near to *Pl. dentata* (KARRER) as described and figured by LIEBUS from the dalmatian Eocene which is probably not the same as *Pl. dentata* (KARRER) described by KARRER from Moravia and Austria. Dimensions: 3.2 × 1.6 × 0.2 mm.

Saracenaria schencki CUSHM. a. HOBSON var. DE WITT PUYT. (Fig. 35, 36).

The specimens differ from S. schencki in lacking the distinct spinose processes and the keel. They might as well be regarded as a variety of S. italica DEFRANCE. Length 1.4 mm.

Spiroplectoides clotho (GRZYBOWSKI).

It is not sure, whether our specimens are arenaceous or non-arenaceous; in the first case they would belong to *Textularia,* in the second to *Spiroplectoides.* Test flattened with nearly parallel sides. Chambers numerous and of equal size in adult. Initial chambers hardly visible, arranged in a flat spiral. Aperture transverse and elliptic. Length 1.3 mm; breadth 0.37 mm.

Bulimina inflata SEGUENZA.

The Cuban specimens have an initial spine.

Uvigerina auberiana D'ORBIGNY. (Fig. 31, 32).

The Cuban specimens lack the initial spine, mentioned in lit. 42, p. 363. They are finely hispid; the short spines being locally arranged in rows.

Uvigerina canariensis D'ORBIGNY, var. NUTTALL. (Fig. 27, 28, 29, 30).

Some specimens agree very well with NUTTALL's variety; other ones differ by the presence of very faint striae covering the whole fossil or by complete smoothness of the test.

Uvigerina havanensis CUSHMAN and BERMUDEZ, var.

At the side of specimens, well agreeing with *U. havanensis* there are much smaller forms with an average length of 0.55 and a breadth of 0.37 mm with many basal spines and built up by fewer chambers, arranged in an elongate spiral, and with sutures not passing from one chamber to the next one. They are nearly related to *U. yazooensis* CUSHMAN.

Pullenia sphaeroides D'ORB., var. *cubensis* n. var. DE WITT PUYT. (Fig. 33).

The only difference with *P. sphaeroides* is the presence of 5 or, rarely, 6 chambers in the last coil.

Eponides byramensis CUSHM., var. *cubensis* PALMER a. BERM. (lit. 68).

According to the authors this variety is almost equally biconvex. In the typefigure, however, the ventral side is much more convex than the dorsal side. Our specimens agree with the typefigure. The limbation is very strong. Ventral umbo usually not filled. Diameter $\leqq$ 1.3 mm, height $\leqq$ 0.8 mm.

Eponides haidingeri (D'ORBIGNY).

According to different authors the number of chambers in the last formed whorl varies

strongly: 6 (lit. 66), 9 (litt. 5), 9 (lit. 59), 9 (lit. 10), 7 (lit. 28) and 5 (lit. 42). Probably this is caused by the indistinctness of the sutures. Our specimens have 9 chambers in the last formed whorl. The dorsal sutures are somewhat raised and limbate. The whole surface is finely perforated. On the ventral side the sutures are invisible. Diameter $\leqq$ 0.9 mm.

Eponides tschoppi n. s. VAN BELLEN (Fig. 10, 11, 12).

Test biconvex. Ventral side somewhat more convex than dorsal side. Periphery rounded, not lobate, not keeled. Dorsal side showing 3 coils. Dorsal sutures limbate, flush with surface, somewhat oblique. Spiral suture limbate, flush with surface. About 8 chambers in last whorl. Ventral sutures indistinct, limbate, flush with surface, last one slightly depressed. Ventral umbo sometimes with area of clear shell material, produced by the meeting of the limbate sutures. Surface perforated. Aperture a slit at ventral side, from periphery to about $^1/_3$ of distance to umbilicus. Diameter 1.10 mm; height 0.44 mm. Differs from *E. praecinctus* in missing the strongly limbate and raised ventral sutures, in having an umbonal mass of clear shellmaterial, in having only 8 chambers in last whorl. Differs from *E. coccaensis* (lit. 23) in being ventrally more convex than dorsally and in having only 8 chambers in last whorl. It may be observed that in lit. 31 the description of *E. cocoaensis* does not agree with the figures: see number of chambers in last whorl, dimension of last chamber, planoconvexity of test.

Rotalia cubensis n. s. VAN BELLEN. (Fig. 13, 14, 15).

Test almost equally biconvex. Dorsal side showing $3^1/_2$ whorls, last one with about 10 chambers. Spiral suture limbate, somewhat raised. Dorsal sutures limbate, flush with surface. Ventral sutures emerging obliquely from the umbo, with a slight curvature backward near the periphery, last two depressed, rest flush with surface. Umbo filled with a knob of clear shellmaterial. Periphery acute with a slight keel, well developed at the beginning of the last whorl and gradually disappearing to the end. Surface coarsely perforated. Aperture between umbo and periphery at ventral side, near to the periphery, but only visible in 2 of the 10 specimens. Diameter $\leqq$ 0.5 mm; height $\leqq$ 0.2 mm. Differs from *R.* cf. *partschiana* WHITE in being equally biconvex and in having more chambers in last whorl. Differs from *Cibicides ungerianus* by its aperture, in missing the typical hispid, dorsal umbo and in being equally biconvex.

Siphonina advena CUSHMAN.

Our specimens agree well with the description and figures in lit. 21. The description in lit. 47 does, however, not at all agree with the figure: see number of chambers in last whorl and diameter. Nor does the general aspect of the figure in lit. 21 agree with that in lit. 47. We have presumed that the descriptions in lit. 21 and lit. 47 and the figures in lit. 21 may be regarded as characteristic for the species. It differs from *S. bradyana* in missing the broad carina and the broad flaring lip, in having a longer neck and depressed sutures on the ventral side. Moreover the keel is not tubulated. It differs from *S. tenuicarinata* by the not-tubulated keel and the shape of the aperture.

Siphonina tenuicarinata CUSHMAN.

There is some variation in diameter of test, in form of aperture, in length of neck and development of keel.

Siphonina reticulata (CZJZEK).

There are two differences with the description: our form has 5 chambers in the last whorl and a not-wholly tubulated keel.

Siphonina tubulosa CUSHMAN.

There is some variation in diameter of test (0.25—0.37 mm) and in number of chambers in the last whorl (4—5). It differs from the foregoing species in having a wholly tubulated keel.

Siphonia cubensis n. sp. VAN BELLEN. (Fig. 1, 2, 3).

Test equally biconvex; periphery subacute; not keeled. Dorsal sutures very indistinct, curved, somewhat limbate, strongly oblique, flush with surface. Ventral sutures somewhat limbate, curved, flush with surface. Last chamber inflated. Aperture a broadly elliptical opening with a well developed neck and phialine lip, on the ventral side, close to the periphery; the axis of the ellipse making an angle with the plane of the periphery. Whole surface finely porous. Diameter $\leqq$ 0.62 mm; height $\leqq$ 0.35 mm. Differs from *S. pulchra* by situation of aperture, by absence of larger pores in central part of ventral face, by more distinct sutures, by smaller dimensions.

Pulvinulinella cubensis n. sp. VAN BELLEN. (Fig. 24, 25, 26).

Test biconvex, ventrally somewhat more than dorsally. Periphery acute, with a broad, thin carina. On dorsal side 2½ whorls visible, last one with about 10 chambers. Dorsal sutures strongly oblique, limbate, very slightly raised or flush with surface. Spiral suture limbate and raised. Ventral sutures oblique, limbate, flush with surface, near periphery slightly depressed. No ventral umbo. Aperture only visible in one of the six specimens. Differs from *P. velascoensis* in missing the strongly elevated sutures between the excavated areas over the dorsal side of chambers, in having limbate, ventral sutures, in missing the ventral umbo.

Anomalina cubensis n. sp. VAN BELLEN. (Fig. 4, 5, 6).

Dorsal side convex, ventral flat. Periphery subacute. Sutures strongly limbate and raised. About 15—16 chambers in last whorl. Dorsal side showing 1½—2 whorls, the first small and covered by secondary shellsubstance, the last lying on a somewhat higher level. Ventral side with depressed umbilicus, in which sometimes small boss of clear shellsubstance. Aperture over the periphery, somewhat extending on dorsal side. Length 0.64 mm; breadth 0.54 mm; height 0.22 mm. Differs from *A. alazanensis* in having a dorsal convex side and a flattened ventral side and in having more limbate sutures. Differs from *A. mecatepecensis* in having not so strongly limbate sutures and in missing the coarsely perforated boss at the dorsal side. Differs from *A. bilateralis* in being planoconvex, in having more limbate sutures, in missing the dorsal umbonal boss.

Planulina cocoaensis CUSHMAN.

Our specimens differ from type specimens in having a mass of clear shellsubstance at dorsal side in stead of small protuberances.

Planulina palmerae n. sp. VAN BELLEN. (Fig. 7, 8, 9).

Our forms agree closely with the description of *P. marielana* by PALMER a. BERMUDEZ (lit. 68), which, however, does not agree with the original description by HADLEY (lit. 51). The differences are:

HADLEY:	PALMER–BERMUDEZ:
9 chambers in last whorl.	12—14 chambers in last whorl.
Ventral umbo a small, slightly depressed area.	Ventral umbo a slightly elevated knob of clear shellsubstance.
Diameter 0.5 mm.	Diameter 1.6 mm.
Dorsal side showing only last whorl, rest hidden by secondary shellsubstance.	Dorsal side showing 2½—3 whorls.
Sutures slightly limbate and elevated.	Sutures heavily limbate and elevated.

The specimen of PALMER and BERMUDEZ has therefore to be renamed; the new name has been given in honour of Mrs. PALMER. In our forms the sense of involution is sometimes clockwise, sometimes anticlockwise.

Planulina mexicana CUSHMAN.

Has more chambers (11) in last whorl than *P. cocoaensis* var. *cooperensis*.

Planulina wuellerstorfi (SCHWAGER).

Is very near to *Cibicides americanus*, the only difference being the raised ventral sutures of *P. wuellerstorfi*; the ventral sutures are depressed in *Cibicides americanus*. Diameter $\leqq$ 1 mm.

Cibicides americanus (CUSHMAN).

Our specimens differ from original description (lit. 11) in having a small ventral umbonal knob. This is also visible in the figure of lit. 26.

Cibicides floridanus (CUSHMAN).

Ventral sutures of our specimens are slightly depressed. There is always a flat border; at about $^1/_6$ of radius begins the convexity of the ventral side. Diameter 0.33—0.84 mm.

Cibicides lobatulus (D'ORBIGNY).

Concave-convex as well as biconvex forms have been brought to this species. Our form agrees closely with the original figure of D'ORBIGNY (lit. 66). It is ventrally convex, dorsally concave. Periphery strongly limbate and irregular. Ventral side showing only last whorl, dorsal side obscured by bad conservation. In last whorl 7 chambers. Ventral sutures depressed and curved; they meet each other in depressed umbilicus. Dorsal sutures depressed. Diameter 0.7 mm.

Cibicides subtenuissimus (NUTTALL).

The bosses not so pronounced as in NUTTALL's typefigures (lit. 59).

Cibicides cubensis n. sp. VAN BELLEN. (Fig. 21, 22, 23).

Test biconvex, dorsal side very slightly, ventral side more strongly so. Subacute keel. Dorsally 1½—2 whorls visible. Dorsal sutures limbate, raised, oblique. Spiral sutures strongly limbate. Surface coarsely perforated. About 9 chambers in last whorl. Ventrally large, central boss of clear shellmaterial. Sutures strongly limbate, flush with surface. Aperture invisible in consequence of small diameter (0.3 mm). Height 0.18 mm. Diameter of ventral boss 0.11 mm. Agrees with *Cibicides spec. A* CUSHMAN in lit. 44.

Cibicides tschoppi n. sp. VAN BELLEN. (Fig. 16, 17, 18, 19, 20).

Dorsal side flattened, ventral side convex. Periphery subacute with thickened keel, which is well developed on early chambers, but gradually disappears on final whorl. Dorsal side only showing last whorl, rest hidden by irregular mass of clear shellmaterial. Dorsal sutures curved; strongly limbate and raised in early part of last whorl, slightly limbate and depressed in last part. Spiral suture strongly limbate. Last whorl with 10—13 chambers. Ventral side somewhat evolute, showing slightly more than 1 whorl. Umbo with boss of clear shellmaterial. Sutures curved, slightly depressed. Surface coarsely perforated. Aperture an arched slit over the periphery. Diameter $\leqq$ 1.05 mm; height $\leqq$ 0.4 mm. Differs from *C. sassei* in having larger size and more chambers in last whorl, from *Eponides cocoaensis* in being planoconvex, in having a dorsal mass of clear shellmaterial, in having limbate dorsal sutures, and in its larger size, from *Planulina wuellerstorfi* in having a dorsal umbonal knob of clear shellmaterial and in its larger size, from *C. pseudoungerianus* in being planoconvex, in having a coarsely perforated ventral side, in having a keel and a not-lobulated periphery.

LITERATURE:

1.W.F.BARBAT and R.E.von ESTORFF. Lower Miocene Foraminifera from the Southern San Joaquin Valley,Cal. J.Pal. 7.164-174.pl.23. (1933).
2.P.J.BERMUDEZ. Estudio micropaleontologico de dos formaciones eocenicas de las cercanias de La Habana. Mem.Soc.Cubana Hist.Nat. 11. 153-180 (1937).
3.---------- Foraminiferos pequeños de las margas eocenicas de Guanajay, Prov. Pinar del Rio. ibid. 11. 319-346.(1937). 12. 1-26 (1938).
4.BORNEMANN. Die mikroskopische Fauna des Septarienthones von Hermsdorf bei Berlin. Z.D.G.G. 7. 307-371v pl.12-21 (1855).
5.H.B.BRADY. Foraminifera. Challenger Report. Zoology. 9. 814. cxv pls (1884).
6.W.S.COLE . A foraminiferal fauna from the Guayabal formation in Mexico. Bull.Amer.Pal. 14, no.51. 48 pp. 5 pls (1927).
7.---------- . A foraminiferal fauna from the Chapapote Formation in Mexico. ibid, 14, no.53. 32 pp. 4 pls (1928).
8.W.S.COLE and P.GILLESPIE. Some small foraminifera from the Meson Formation of Mexico. ibid. 15, no.57B. 19 pp. 4 pls (1930).
9.H.N.CORYELL and J.R.EMBICH. The Tranquilla Shale (Upper Eocene) of Panama and its foraminiferal fauna. J.Pal. 11. 289-305. pls.xli-xliii. 1 fig.(1937).
10.J.A.CUSHMAN.A Monograph of the Foraminifera of the North Pacific Ocean. Bull.U.S.Nat.Mus. 71.5. 81 pp. 31 pls (1915).
11.---------- Some Miocene and Pliocene Foraminifera of the Coastal Plain of the United States. Bull.U.S.Geol.Surv. 676. 100 pp. 31 pls (1918).
12.---------- The smaller fossil Foraminifera of the Panama Canal Zone. Bull.U.S.Nat.Museum. 103. 45-87. pls.xix-xxxiii (1918).
13.---------- Fossil Foraminifera from the West Indies. Contr.to the Geol.a.Pal. of the West Indies. Publ.Carn.Inst.291. 21-71. xv pls (1919).
14.---------- The Foraminifera of the Atlantic Ocean. Bull.U.S.Nat.Mus. 104 part 4. 228 pp. 41 pls (1923).
15.---------- The Foraminifera of the Vicksburg Group. U.S.Geol.Surv. Profess.Paper 133. 11-71. pls i-viii (1923).
16.---------- New Foraminifera from the Upper Eocene of Mexico. Contr.Cushm.Labor.Foram.Research. 1. 4-8. 1 pl (1925).
17.---------- Some fossil Bolivinas from Mexico. ibid, 1. 81-85. pl.12 (1926).
18.---------- Some new Foraminifera from the Upper Eocene of the S.E. coastal plain of the Un.St. ibid. 2. 29-38. pl.4,5 (1926).
19.---------- New and interesting Foraminifera from Mexico and Texas. ibid. 3. 111-119. pl. 32,33 (1927).
20.---------- Some characteristic Mexican fossil Foraminifera. J.Pal. 1. 147-172. pls.23-28 (1927).
21.---------- Foraminifera of the genus Siphonina and related genera. Bull.U.S.Nat.Mus. 72,no.20. 15 pp. 4 pls (1927).
22.---------- Recent Foraminifera from the West Coast of America. Bull.Scripps Instit.Oceanogr. Techn.Ser. 1.10. 119-188. pls.1-6 (1927).
23.---------- Additional Foraminifera from the Upper Eocene of Alabama. Contr.Cushm.Lab.Foram.Research. 4. 73-79. pl-10 (1928).
24.---------- A late tertiary fauna of Venezuela and other related regions. ibid. 5. 77-101. pls 12-14 (1929).
25.---------- Planulina ariminensis d'Orb. and P. wuellerstorfi (Schwager). ibid. 5. 102-105. pl-15 (1929).
26.---------- The Foraminifera of the Choctawhatchee Formation of Florida. Bull.Florida State Geol.Surv. 4. 5-89. pls 1-12 (1930).
27.---------- The Foraminifera of the Atlantic Ocean. Bull.U.S.Nat.Mus. 104, part 8. 178 pp. 26 pls (1931).
28.---------- A preliminary report on the Foraminifera of Tennessee. Bull.Geol.Dept Tennessee. 41. 112 pp. 13 pls (1931).
29.---------- Notes on the Foraminifera described by Batsch in 1791. Contr.Cushm.Labor.Foram.Research. 7. 62-72. pls 8,9 (1931).
30.---------- New Foraminifera from the Upper Jackson Eocene of the S.E.coastal plain of the United States. ibid. 9. 1-12. pls 1,2 (1933).
31.---------- Upper eocene Foraminifera of the S.E. United States. U.S.Geol.Surv. Profess.Paper 181. 88 pp. 23 pls (1935).
32.---------- A monograph of the foraminiferal family Verneuilinidae. Cushm.Labor.Foram.Research. Spec.Publ. 7. 157 pp. 20 pls (1937).
33.---------- A monograph of the foraminiferal family Valvulinidae. ibid. no.8. 210 pp. 24 pls (1937).
34.---------- A monograph of the foraminiferal subfamily Virgulininae. ibid. no.9. 228 pp. 24 pls (1937).
35.---------- A monograph of the foraminiferal family Nonionidae. U.S.Geol.Surv. Prof.Paper 191. 91 pp. 20 pls (1939).
36.J.A.CUSHMAN and P.J.BERMUDEZ. Additional new species of Foraminifera and a new genus from the Eocene of Cuba. Contr.Cushm.Labor.Foram.Research. 12. 55-63. pl.10,11 (1936).
37.---------- Further new species of Foraminifera from the Eocene of Cuba. ibid.13. 1-29. pl.1,2. (1937).
38.J.A.CUSHMAN and E.D.CAHILL. Miocene Foraminifera of the coastal plain of the Eastern United States.U.S.Geol.Surv.Prof Paper.175A.1-50.13 pls (1933).
39.J.A.CUSHMAN and H.H.BECK. A foraminiferal faunule from the Type San Lorenzo Formation,Santa Cruz Cy.Cal.Contr.Cushm Labor.Foram.Research.11.53-62.pl.8,9 (1935).
40.J.A.CUSHMAN and P.W.JARVIS. Cretaceous Foraminifera from Trinidad.ibid. 4. 85-103. pl.12-14 (1928).
41.---------- Miocene Foraminifera from Buff Bay,Jamaica.J.Pal. 4. 353-368. pl.32-34 (1930).
42.---------- Upper Cretaceous Foraminifera from Trinidad. Proc.U.S.Nat.Mus. 80. 14. 60 pp. 16 pls (1932).
43.J.A.CUSHMAN and B.LAIMING. Miocene Foraminifera from Los Sauces Creek,Ventura Cy.Cal. J.Pal. 5. 79-120. pl.9-14. 5 figs (1931).
44.J.A.CUSHMAN and J.H.PEASTERS. Middle eocene Foraminifera from the Llajas formation,Ventura Cy.Cal. ibid.10. 497-517.pls.74-77-figs 1-4 (1936).
45.J.A.CUSHMAN and F.A.PARKER. Miocene Foraminifera from the Temblor formation of the E.side of the San Joaquin valley.Cal. Contr.Cushm.Labor. Foram.Research.7. 1-16. pls.1,2. (1931).
46.J.A.CUSHMAN, R.E. and K.C.STEWART. Tertiary Foraminifera from Humboldt Cy. Cal.Trans.San Diego Soc.Nat.Hist. 6. 44-94. 8 pls (1930).
47.J.A.CUSHMAN and C.WYTHE COOKE. The Byram calcareous marl of Mississippi and its Foraminifera.U.S.Geol.Surv. Profess.Paper 129E. 79-122. pls.14-28 (1922).
48.J.CZIZEK. Beitrag zur Kenntnis der fossilen Foraminiferen des Wiener Beckens. Haidingers Naturw.Abh. 2.1. 137-150. pls.12,13 (1848).
49.J.P.DORR. New data on the correlation of the Lower Oligocene of S. and C.America with that of S.Mexico. J.Pal. 7. 432-438 (1933).
50.J.J.GALLOWAY and M.MORREY. A lower tertiary fauna from Manabi,Ecuador. Bull.Amer.Pal. 15. 55. 56 pp. 6 pls (1929).
51.W.H.HADLEY. Some tertiary Foraminifera from the north coast of Cuba. ibid. 20. 70a. 40 pp. 5 pls (1934).
52.H.J.HEDBERG. Foraminifera of the Middle tertiary Carapita formation of northeastern Venezuela. J.Pal. 11. 661-697. pls.90-92 (1937).
53.R.V.HOAG. Addition to the list of species occurring in the type Red Bluff Clay,Hiwannee,Mississippi. J.Pal.2. 173-176 (1928).
54.H.V.HOWE and W.P.MALLORY. Foraminifera of the Jackson Eocene at.....,Louisiana. Louisiana Geol.Bull. 2. 118 pp. 15 pls. 2 figs (1932).
55.F.KARRER. Zur Foraminiferenfauna in Österreich.Sitz.ber.K.Akad.Wiss.Wien. Math.Natw.Cl. 55. 331-368. pl.1-3 (1867).
56.A.LIEBUS. Die Tertiarformation in Albanien.Die Foraminiferen. Palaeontographica. 70. 41-114 (1928).
57.H.W.MATTHES. Die Lagenen des deutschen Tertiärs. Palaeontographica. 90.Abt.A. 49-108. pls.3-8 (1939).
58.W.L.F.NUTTALL.Notes on the Tertiary Foraminifera of Southern Mexico.J.Pal. 2. 372-376. pl.50. (1928).
59.---------- Tertiary Foraminifera from the Naparima Region,Trinidad (B.W.I.).Q.J.G.Soc.London. 84. 57-115. pls.3-8 (1928).
60.---------- Eocene Foraminifera from Mexico. J.Pal. 4. 271-293. pls. 23-25. 1 textfig. (1930).
61.---------- Lower Oligocene Foraminifera from Mexico. ibid. 6. 3-35. pls.1-11. (1932).
62.---------- Upper Eocene Foraminifera from Venezuela.ibid. 9. 121-131. pls.14,15. (1935).
63.A.d'ORBIGNY. Tableau méthodique de la classe des Céphalopodes.Ann.Sc.Nat. 7. 96-169,245-314. pls.10-17 (1826).
64.---------- Foraminifères. In: R.de la SAGRA.Histoire physique....de l'île de Cuba. 224 pp .Atlas. (1839).
65.---------- Foraminifères. In: Voyages Amérique méridionale. 5. pt.5. 1-77. pls.1-9 (1839).
66.---------- Foraminifères fossiles du bassin de Vienne (Autriche). 132 pp. 21 pls (1846).
67.D.K.PALMER. The foraminiferal genus Gumbelina in the Tertiary of Cuba.Mem.Soc.Cubana Hist.Nat.8. 73-76. figs.1-8 (1934).
68.D.K.PALMER and P.J.BERMUDEZ. An oligocene foraminiferal fauna from Cuba. Mem.Soc.Cub.Hist.Nat. 10. 227-316. pls.13-20 (1936).
69.H.A.PIJPERS. Geology and palaeontology of Bonaire (D.W.I.).Geogr.Geol.Meded.Utrecht. Physiogr.Geol.Reeks.8. 103 pp. (1933).
70.---------- Ruttenia, a new name for Bonairea Pijpers. Contr.Cushm.Labor.Foram.Research. 9. 30 (1933).
71.A.E.REUSS. Beiträge zur Kenntnis der tertiären Foraminiferen-Fauna.Sitz.Ber.Akad.Wiss.Wien.Math.Natw.Classe.48. 36-71.pls.1-8 (1863).
72.C.SCHWAGER. Fossile Foraminiferen von Kar Nikobar. Reise der.....Novara.Geol.Theil. 11.2. 187-268. pls.4-7 (1866).
73.R.J. and K.P.STEWART. Post-miocene Foraminifera from the Ventura Quadrangle, Ventura County, Cal. J.Pal.4. 60-72. pls.8,9 (1930).
74.M.TERQUEM. Les Foraminifères de l'Eocène des environs de Paris. Mém.Soc.Géol.de France. 2.3. 193 pp. pls 9-27 (1882).
75.H.B.THALMANN. Nomenclator zu den Tafeln 1 bis 115 in H.B.BRADY's Werk über die Foraminiferen der Challenger-Expedition,London 1884. Ecl.Geol.Helvetiae. 25. 293-312 (1932).
76.---------- Nachtrag zum Nomenklator zu BRADY's Tafelband der Foraminiferen der Challenger Expedition. Ecl.Geol.Helvetiae. 26. 251-255 (1933).
77.A.A.THIADENS.Cretaceous and tertiary Foraminifera from southern Santa Clara Province,Cuba. J.Pal. 11. 91-109. pls.15-19 (1937).
78.W.P.WHITE. Some index Foraminifera of the Tampico Embayment of Mexico. ibid. 3. 30-57. pls. 4,5. (1929).
79.J.P.C.de WITT PUYT. Geologieche u.palaeontol.Beschreibung der Umgebung von Ljubuški,Hercegovina.Acad.Thesis.Utrecht.99 pp. 5 pls.Map.Sections.(1941).

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38

LIST OF SPECIES	2353	2354	2359	2362	2363	2292	2293	2294	2295	2296	2297	2298	2299	2621	2622	2695	2714	2720	3050	3051	3052	3054	3055	3056	3058	3059	3063	3071	3081	3303	3321	3323	3490	3530	BIBLIOGRAPHY	UPPER CRETACEOUS	EOCENE	OLIGOCENE	MIOCENE	PLIOCENE	RECENT
1.Textularia mississppiensis Cushman var.alazanensis Nuttall																					x														61,68.			xCM			
2.Vulvulina pennatula (Batsch)	x		x				x	x										x				x	x	x			x						x	5,51,59.		xT	xC			
3.Gaudryina cubana Cusm.a.Bermudez				x												x	x				x				x									32,36.		xC				
4.--------- jacksonensis Cushman									x					x			x	x							x								x	18,32,61,68.		xG	xCM			
5.--------- trinitatensis Nuttall	x								x												x				x	x								24,32,59,68.		xT	xC			
6.Clavulinoides marielinus Cushm.a.Berm.										x									x			x	x	x			x							32,37.		xC				
7.---------- cubensis Cushm.a.Berm.	x							x									x				x		x		x				x	x				32,37.		xC				
8.---------- havanensis Cushm.a.Berm.																													x						32,37.		xC				
9.Clavulina angularis d'Orb.																x																			33.						
10.Dorothia cylindrica (Nuttall)																							x	x	x				x						33,61.			xM			
11.Karreriella chapapotensis (Cole)																												x					x		6,33,60.		xM				
12.Listerella venezuelana (Nuttall)										x				x																					33,62.		xV				
13.Cristellaria kemperi Hanna														x																					61.			xM			
14.---------- peregrina Schwager var. matanzana Palmer a.Bermudez							x			x																x		x							68.			xC			
15.---------- (R) articulata (Reuss)																				x						x									2,6,59,68.			xCM	xC		
16.---------- (R) calcar (L)																	x				x							x							5,24,41,51,59.				xC	xHJT	x
17.---------- (R) clericii Fornasini														x																					24,59.					xTV	
18.---------- (R) cultrata (Montf)	x	x		x	x		x			x	x			x	x	x	x	x	x	x			x		x			x			x		x	20,59,60,61,68,69.			xBM	xCM	xHJ	x x
19.---------- (R) formosa Cushman														x			x			x				x				x	x				x	5,41,51,59,61,68.				xCM	xJT	x x
20.---------- (R) iota Cushman	x			x										x				x	x	x			x		x	x				x	x			5,14,38,68.				xC	xG	x x
21.---------- (R) lucida Cushman				x										x									x			x								5,14,						x x
22.---------- (R) orbicularis (d'Orb.)	x																x			x														5.						x x
23.---------- (R) rotulata (Lamarck)			x							x				x				x	x		x		x	x	x			x				x	x	3,8,59,60.	xAM	xMC	xC	xJT		x x
24.---------- (L) convergens Bornemann				x		x	x			x							x	x	x	x		x										x	2,3,5,7,8,20,60,68.		xCM	xCM			x x
25.---------- (L) crassa d'Orbigny			x							x				x	x								x										5,68.			xC			x x
26.---------- (L) gibba d'Orbigny						x	x								x		x		x						x				x	x				5,14,59.				xT		x x
27.---------- (L) subpapillosa Nuttall	x	x													x		x		x		x			x	x		x							51,61.			xCM			
28.Planularia danvillensis Howe a.Wallace																										x										xCG	xC				
29.---------- brandsi nov.sp. de Witt Puyt																										x															
30.---------- thiadensi nov.sp. de Witt Puyt																							x															xC				
31.---------- westermanni Pijpers										x										x						x			x						69.			xB				
32.---------- spec.								x																																			
33.Marginulina crepidula (F.a.M.)			x				x	x	x	x			x					x										x				x			5,59,61,60.	xM	xM	xM	xT		x
34.---------- subaculeata(Cushm) var.glabrata(Cushm)		x		x													x		x		x				x							x		24,59.				xTV		
35.---------- sublitua (Nuttall)								x		x																									61,68.			xCM			
36.Dentalina cocoaensis (Cushman)	x			x																															16,20.		xM	xM			
37.---------- semilaevis Hantken										x																x	x								2,3,61,68.		xC	xCM			
38.Nodosaria acuminata Hantken																																			20,24,61,68,69.		xC	xCM			
39.---------- aff.carinaba(Neugeboren)		x	x		x										x					x							x							61.			xM			
40.---------- erbessina Schwager	x																																		51,59.			xC	xT		
41.---------- ewaldi Reuss								x																						x					20.		xG	xM			
42.---------- halkyardi(Cushm) var.antillana Palm.a.Berm.																																	x		30,68.		xE	xC			
43.---------- hispida d'Orb.			x																																5,59.				xT		
44.---------- matanzana Palmer a.Bermudez							x	x																											68.			xC			
45.---------- obliquata (Batsch)								x		x																		x							29,51.			xC			
46.---------- raphanistrum (L)														x																					5,12,13,59,61,68.		xP	xCM	xT		x
47.---------- schlichti Reuss								x																											24,59.				xT		
48.---------- spinicosta d'Orb.										x						x																			2,20,61,66,68.		xC	xCM			
49.---------- sp.cf.Nodosaria mexicana Cushman																								x											6,20,51,60.						
50.Pseudoglandulina comatula (Cushman)		x																		x								x							2,14,59,61,68.		xC	xCM	xT		x
51.Saracenaria italica Defrance														x							x														3,5,24,59,61,68.			xCM	xJT		x x
52.---------- latifrons (Brady)								x																											10.						x
53.---------- schencki Cushm.a.Hobson	x	x						x												x								x							39,68.			xC			
54.---------- schencki Cushm.a.Hobson var.	x	x				x	x			x										x								x	x												
55.---------- senni Hedberg	x	x	x											x			x													x	x				52.			xCV			
56.Vaginulina elegans d'Orb. var.mexicana Nuttall																					x			x									x		3,61,68.			xCM			
57. Lagena hispida Reuss var.			x				x	x	x	x											x				x	x															
58. ------ laevis (Montagu)						x	x							x																					5,14,69.		xB		xT		x
59. ------ marginata (Walk.a.B.)							x			x																									5,14,69.	xM	xB				x
60.------- orbignyana Seguenza							x	x					x															x							5,14,68,69.		xB	xC	xT		x x
61.------- striato-punctata Parker a.Jones							x																												2,5,14.		xC				x
62.------- sulcata (Walk.a.Jac.) var.strumosa Reuss	x			x		x			x	x	x				x	x					x					x			x		x	x			12,68.			xC			
63.Nonion mexicanum Cole																																			6.		xGM				
64.------- pompilioides (F.a.M.)	x	x				x	x			x	x					x			x				x					x			x	x			5,24,59,61.			xCM	xTV		x
65.Operculina sp.	x																																								
66.Spiroplectoides ?clotho (Grzybowski)																								x											20,40,42,78.	xT					
67.Gumbelina cubensis Palmer												x														x	x					x			67,68.			xC			
68.Plectofrondicularia vaughani Cushman		x					x		x	x																									2,20,24,9,41,51,68.		xCP	xCM	mJV		
69.Bulimina inflata Seguenza		x					x																												6,9,24,41,51,59,61,68,78.		xMP	xCM	mJTV		
70.Bolivina alazanensis Cushman							x	x	x	x	x	x		x	x	x		x							x				x	x	x	x			17,20,34,61,68.		xGM	xCM			
71.--------- arta Macfadyen																					x	x													34.						
72.--------- malkinae Coryell a.Embich					x											x																			9.		xP				
73.--------- mantaensis Cushman																											x	x							24,34.				xEV		
74.--------- matanzana Palmer a.Bermudez																	x																		34,68.			xC			
75.--------- mexicana Cushm.var.aliformis Cushman								x																		x				x	x				17,20,34,51,61,68.		xM	xCM			
76.--------- pisciformis Gall.a.Morrey																																			24,34,50,68.			xC	?E		
77.Uvigerina asperula Czjzek	x																																		5.						x
78.---------- auberiana d'Orb.													x											x											24,41,59,61,64.			xM	xJV		x
79.---------- canariensis d'Orb. var.Nuttall		x	x	x																															61,68.			xCM			
80.---------- cubana Palmer a.Bermudez																	x	x																	68.			xC			
81.---------- havanensis Cushm.a.Bermudez																																			36.		xC				
82.---------- pygmaea d'Orb.,var. Nuttall							x		x	x	x				x	x																	x		59.				mT		
83.Siphogenerina transversa Cushman				x					x											x	x								x						12,51,59.		xT	xCP			
84.---------- transversa, var.cubensis C.a.Berm.	x	x	x	x																x	x			x	x				x	x					37.		xC				
85.Nodosarella paucistriata Gall.a.Morrey																												x							24,61.			xM	xV		
86.---------- subnodosa (Guppy)																											x	x					x		3,2,7,59,61,68.		xCM	xCM	xT		
87.---------- spec.,cf. nov.spec.Cushman																												x													
88.Ellipsonodosaria tuckerae Hadley										x																									2,3,51.		xC	xC			
89.---------- verneuili (d'Orb)			x								x									x												x			2,3,24,41,51,66,68.		xC	xC	xJTV		
90.Valvulineria jacksonensis Cushman							x			x																									30,31.		xG				
91.Gyroidina soldanii d'Orbigny																												x							5,10,12,39,50,61,62,63,73		xVE	xM	xG		
92.Eponides affinis (Czjzek)							x								x	x	x												x						43,48,61,62.		xV	xM			
93.---------- ?byramensis (Cushman)																											x								8,47,67,69.		xB	xCGM			
94.---------- byramensis Cushm.,var.cubensis P.a.Berm.	x			x																	x			x		x								68.			xC			
95.--------- haidingeri (d'Orb.)																			x						x										5,10,12,28,42,59.	x			xT		x
96.--------- praecinctus (Karrer)	x																																		5,10,13,15.						x
97.--------- pygmaeus (Hantken)	x																																		5,10,12,27,31,60.		xGM				x
98.--------- tschoppi nov.sp.Van Bellen			x								x											x	x	x	x															
99.--------- umbonatus(Reuss)	x			x		x			x	x	x											x	x		x			x				x			5,7,10,31,50.		xGM				x
100.Rotalia cubensis nov.sp.Van Bellen																		x				x																			
101.Siphonina advena Cushman															x																				8,21,47.			xGM			
102.---------- cubensis nov.sp.Van Bellen			x																																						
103.---------- reticulata (Czjzek)																							x												21,48.						
104.---------- tenuicarinata Cushman	x	x				x	x	x		x	x	x	x			x	x	x	x	x		x		x		x	x	x					x		6,9,20,21,24,51,61,68.		xMP	xCM			
105.---------- tubulosa Cushman	x						x			x							x									x									21.						x
106.Amphistegina spec.																			x									x													
107.Pulvinulinella cubensis nov.spec.Van Bellen							x			x																				x											
108.Pullenia quinqueloba(Reuss)											x																								6,42,59,60,61,68.	xT	xCM	xCM	xT		
109.--------- sphaeroides (d'Orbigny)																											x								3,5,59,60,61,68.	xM	xCM	xCM	xT		x
110.--------- sphaeoides,var.cubensis n.v.de Witt Puyt										x	x			x																											
111.Globigerina apertura Cushman				x			x		x																										2,5,6,8,11.		xM	xMC	xG		x
112.---------- bulloides d'Orbigny							x		x										x		x			x	x			x					x		5,7,8,13,59,61,66,69.	x	xBM		xJP		x
113.---------- ?bulloides d'Orbigny																				x	x			x	x																
114.---------- bulloides d'Orb.,var.triloba Reuss	x																									x									5,59.				xT		x
115.---------- socaena Gumbel	x	x				x														x															5,7.			xM			
116.---------- socaenica Terquem	x			x																		x													7,8,74.			xM	xM		
117.----------?socaenica Terquem		x					x		x												x			x																	
118.Anomalina alazanensis Nuttall											x					x																			61,68.				xCM		
119.---------- cubensis nov.sp.Van Bellen										x						x																									
120.Ruttenia coronaeformis (Pijpers)																			x																69,70.		xB				
121.Planulina alazanensis Nuttall																			x				x												61.			xM			
122.---------- cocoaensis Cushman																													x						23,31.		xG				
123.---------- cocoaensis Cushman,var.coöperensis C.																x																			30,31.		xG				
124.---------- mexicana Cushman																	x																			19,20,49,51,61,68,69.		xB	xCM			
125.---------- ornata d'Orbigny						x				x							x																			22,46.					x	
126.---------- palmerae nov.sp.Van Bellen										x							x																									
127.---------- wuellerstorfi (Schwager)			x			x				x		x							x	x			x						x	x					5,10,11,25,27,50,59,72.		xE		xTG		x
128.Laticarinina pauperata (Parker a.Jones).		x	x				x	x		x	x	x							x		x								x					5,27.						x
129.Cibicides akneranus (d'Orbigny)															x					x															5,61,66.			xM			x
130.---------- americanus (Cushman)				x		x														x	x						x								8,11,12,26,38,47,53,58,61			xGM	xGT		
131.---------- americanus,var.crassiseptus C.a.L.							x				x												x												43.						
132.---------- cubensis nov.sp.Van Bellen										x														x																		
133.---------- floridanus (Cushman)																			x	x			x	x	x	x									11,26,38,43,45,58,59.				xGT		
134.---------- lobatulus (W.a.J.)							x																												5,8,10,11,27,31,47,59,74.		xG	xG	xG	x	x
135.---------- mexicanus Nuttall							x	x		x						x																			51,61.			xCM			
136.---------- pseudoungerianus (Cushman)																						x													5,8,9,1231,45,75.		xG	xGM			x
137.---------- subteniussimus (Nuttall)			x							x					x	x	x	x	x										x					59,68.			xC	xT		
138.---------- trinitatensis (Nuttall)	x														x	x	x	x	x				x							x	x				51,59,61,68.			xCM	xT		
139.---------- tschoppi nov.spec.Van Bellen										x			x																												
140.---------- ungerianus (d'Orbigny)																										x									50,61,62,66.		xE	xM			

Significance of figures in the stratigraphic columns: A=Antigua B=Bonaire C=Cuba E=Ecuador G=Southern U.S.A. H=Haiti J=Jamaica M=Mexico P=Panama T=Trinidad V=Venezuela.

Zoology. — *Note on a diseased dental condition in Hippopotamus amphibius L.* By D. A. HOOIJER. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of October 25, 1941.)

An adult skull of *Hippopotamus amphibius* L. from Inhamissengo, Mozambique (cat. ost. n, don. H. P. N. MULLER, 1893) in the collection of the Rijksmuseum van Natuurlijke Historie at Leiden shows a remarkable abnormality in the arrangement of the lower incisors. Normally in hippopotami four incisors develop in the lower jaw. The cylindrical inner ones are the largest, their alveoli extend almost throughout the length of the symphysis. The incisors of the outer pair are smaller; they are laterally compressed and their upper margins are placed in a straight line with, or a little higher than those of the central ones. The depth of their alveoli varies from 0.45 to 0.60 of those of I_1.

In our specimen (fig. 1), instead of two large central incisors, there is on the right side, considerably behind the level of the outer incisors (the left was removed before the picture was taken) a peculiarly shaped alveolus. The tooth is lost, the alveolus is dorso-ventrally compressed. the horizontal diameter being 38 mm, the vertical only 28 mm. The depth is 70 mm, whilst the length of the symphysis measures 165 mm! The alveolus of this right I_1 does not run parallel to those of the two I_2, but has an oblique direction, the posterior part is situated above that of the right I_2 and below P_3. As a result of this position of I_1 the anterior root of P_2 (P_1 does not occur in the permanent set) is partially absorbed and bent posteriorly so that the tooth can be easily removed from the jaw. P_3, of which only the roots are present, has undergone a rotation as can be seen in the photograph, causing a broadening of the jaw, which has, to a smaller extent, also occurred in the corresponding part of the left ramus. The presence of a thin bony scale, which has been formed inside the alveolus of the right I_1 and which lies on its anterior wall (fig. 3, *a*), may also be taken as proof of the shifting of this incisor to a more posterior position. This bony scale shows a ridge which brought about a longitudinal groove in the incisor.

At the left side, corresponding in level with the alveolus of the right I_1, only a small opening, 11 mm in diameter, is present.

Two longitudinal sections were made through the symphysis, one through the middle of this opening, exactly halfway between the left I_2 and the median plane and the second on a corresponding level at the right side. In normal jaws these sections should have passed longitudinally through the deep alveoli of the central incisors. Here at the left side indeed a cavity was found, resembling an alveolus as far as concerns its situation but much narrower, with a rough wall and at the anterior side passing into a number of narrow channels, one of which is seen in the section (fig. 2). In the cavity, and partly in an expansion of it, there lies an irregular mass of calcified dental tissue (*c*). This calcified mass has locally widened the cavity and has even caused a slight constriction of the alveolus of the adjacent incisor, so that the proximal extremity of the latter is more or less indented laterally (fig. 4, upper figure, lower side at the left). The cavity communicates with the alveolus of the left I_2 by several small openings, whilst it opens to the upper surface of the symphysis by a round hole (*b*), the latter being the only distinct indication of this cavity at the outside of the jaw. The wall of the round hole is smooth in contradistinction to that of the cavity. In the upper part of the symphysis the layer of compact bone is very thick (*e*), in the lower part it has the normal thickness. Under the posterior part of the cavity the bony structure of the jaw is coarse and irregular, whilst next to this coarse part there is a region with finely cancellated bony structure (*d*).

In the right section (fig. 3), the central part shows the same coarse and irregular structure whilst in the lower part there is a cancellous region as in the left section. The solid and very hard exterior wall here too is very thick at the upper and posterior regions. The bony scale, which has already been referred to above, is shown in this section (*a*).

At first one is inclined to regard the cavity at the left side of the jaw as a dentigerous cyst, the fluid contents of which might have found an outlet through the round hole at the upper surface (*b*), this being the spot of least resistance. But against this supposition the following facts may be put forward:

The elongated shape of the cavity is quite similar to that found normally in very young animals, where it contains the germ of the permanent incisor. When cysts occur, they grow to a rounded shape, and in further development they cause an expansion of the jaw. Moreover the round hole has a smooth wall, whilst outlets of pus always are much smaller and have a more or less rough wall.

Therefore this hole must be regarded as the alveolus of the deciduous incisor.

The narrow and smooth channels, which run forward from the cavity are those which are present in normal jaws too, in the living animal they contain bloodvessels. They are always found between the teeth and especially in the frontal part of the lower jaw.

As the alveolus of the left first deciduous incisor corresponds in place exactly with the opening of the alveolus of the right central incisor it is probable that the right deciduous incisor also erupted in this uncommon spot.

For an explanation of the origin and cause of this abnormality two possibilities must be considered.

I. The incisors might have shifted to an uncommon position as a result of a narrowing of the jaw. This phenomenon has been met with in several species of mammals. In the present case, however, this explanation does not hold, for:

1. the jaw in this specimen is not narrower, but even broader than in some normal specimens, and

2. the teeth have shifted over a much greater distance than is usually caused by a narrowing of the jaw.

II. The deciduous incisors might have shifted to this abnormal position already before erupting. Neither COLYER (1936) nor BENNEJEANT (1936) mention cases of misdirection of the eruption of deciduous teeth, but this does not mean that it is impossible. The milk dentition in *Hippopotamus amphibius* L. is not long in use. SIGEL (1883, p. 11), e.g., records that the change of the lower central incisors occurs at the age of ca. 1¼ year, while the maximum duration of life (in zoological gardens; data from animals in the wild state are not known) recorded is 41 years (FLOWER, 1931, p. 225).

As Mr. C. N. VAN RIJSWIJK kindly informed me, abnormal eruption of deciduous teeth in man, due to trauma by delivery by forceps, has been recorded by HENRY A. P. WILD at the 53rd meeting of the American Dental Society of Europe at Lausanne, August 7—9, 1939. The latter cause seems to me the most probable, and, therefore, I believe that the present case must be explained as follows:

As a result of trauma in this exposed and vulnerable part of the jaw during or shortly after birth the germs of the central incisors may have become displaced and afterwards erupted behind the level of the outer incisors [1]). These milk incisors may have become carious, perhaps also on account of their abnormal situation, and thereupon pulpitis and periodontitis originated in the left incisor. The rough wall of the cavity indicates the action

---

[1]) KERBERT records that in the twenty-five cases observed in the zoological garden at Amsterdam up to 1922 the birth always took place on land, though the animal was given free access to the water (KERBERT, 1922, p. 189). Dr. A. L. J. SUNIER kindly informed me that also in later years at Amsterdam it never occurred otherwise. Several cases are known, however, in which the animal was born in the water, e.g., that recorded by ALVING (1932). VEVERS (1926) assumes that in nature the young probably is born in a bed of reeds close to the edge of the water or in very shallow water.

of pus bacteria. The germ of the left central incisor became surrounded with pus and devitalised; the disease, which must have affected the animal at an early stage after the eruption of the deciduous incisors then developed into a chronic ostitis and may have caused the displacement of the right I_1, which for the rest has followed his predecessor and erupted, to a still more posterior position. In the meantime the ostitis has given rise to an eburnification in the upper section of the jaw, on the other hand to a rarefaction of the bony tissue in the lower section of the jaw. The small alveolus at the left side may have been filled with tissue, or, as is not uncommon when the eruption of the permanent tooth is prevented, the deciduous incisor may have subsisted for a long time; so the pus could not reach the small alveolus and its wall remained smooth.

As the animal is fully adult and no external expansion of perforations of the jaw are visible, it was only a local inflammation. The rest of the skull shows no morbid alterations except for a carious upper left fourth premolar. The disease was still progressing at the time when the animal was shot and could have given rise to osteomyelitis.

The importance of the diagnosis of this anomaly in the dentition of *Hippopotamus amphibius* L. is evident as the determination of the fossil hippopotami from the Siwaliks and Java for the greater part rests upon minor differences in the arrangement of the lower incisors.

Due to the endostomatic situation of the lower central incisors the outer incisors, especially the left one, are placed nearer to the mesial line and are larger than normally. The ratio of the distance separating the outer incisors (86 mm) to that between the lower canini (240 mm) is 0.36, while in normal specimens this ratio varies from 0.44 to 0.60. The depth of the alveolus of the outer incisor at the right side is 110 mm, at the left side 115 mm. In normal cases the corresponding central incisors would have had alveoli of at least 200 mm depth, the length of the symphysis in our specimen measures only 165 mm. The left outer incisor (upper tooth of fig. 4) with its indentation at the pulp cavity, is twisted for about 90°. Its situation at the pulp cavity is normal, the vertical diameter being the largest, while at the alveolar border the greatest diameter is placed horizontally. The obliquely abraded surface at the outer part of the extremities of both I_2 is more pronounced than in normal specimens, in which the outer incisors oppose themselves more directly to the outer pair in the upper jaw and at times even are worn towards the inner side of their tips. The greatest diameter at the right side is 30 mm, at the left side 33 mm. In normal adult skulls with lower outer incisors from that size, the central incisors have a diameter of 50 mm or more!

Due to the shifting of the lower I_2 nearer to the mesial line the upper I_2 are slightly bent inwards. The narrow strip of enamel along the convex side of these upper teeth has cut a deep groove in the lower incisors (fig. 4).

The upper central incisors show no trace of friction with antagonists, the displaced right lower I_1, therefore, did not come into contact with one of them. The left is shorter than the right, their tips are rounded off like those of the tusks of an elephant, probably caused by rubbing against hard objects. The left on its posterior surface shows a horizontal so-called wedge-shaped defect (fig. 5), which cannot be explained as having originated by friction against an antagonist. Similar wedge-shaped defects have been recorded as occurring on the incisors and canini of horses, ruminants and Kamchatka-bears (BREUER, 1933, p. 113); like the former the latter feed on grass. The defect in our specimen is only shallow and cannot be artificial for it has a brown colour like undamaged parts of the tooth.

BREUER, who describes wedge-shaped defects on the posterior surface of the lower canini of *Ursus spelaeus* Rosenm.[1]) suggests that they are caused by mechanic erosion when tearing off grass (l.c., p. 106). Although BREUER's statements apply to defects on the lower teeth exclusively, in the present case the following explanation may hold:

---

[1]) KUBACSKA (1935) found them too on the external surface of the lower I_3 opposing the canine, and on the upper canini of the same species (l.c., pl. V, fig. 3, 7).

Due to their endostomatic situation the central lower incisors could not perform their normal function while grazing and uprooting aquatic plants; the upper incisors took over the function of the lower ones. Consequently the vertical pulling movement was changed into a movement directed sideways to the right, and the grassblades and stalks were pulled off between the upper teeth. On the anterior surface of I_2, however, erosion was entirely prevented by its strip of enamel (fig. 5). The defect on the posterior surface of the first incisor is only shallow because the longitudinal strip of enamel along its outer surface more or less prevented the erosion.

The fact that the free part of the left upper I_1 is shorter than the right indicates that it was more in use. As consequently the animal moved its head to the right when feeding in all probability it was left-handed.

For valuable information and kind advice I am greatly indebted to Dr. L. D. BRONGERSMA, Leiden; Dr. TH. E. DE JONGE, Amsterdam; Mr. C. N. VAN RIJSWIJK, D. D. S., The Hague; Dr. A. L. J. SUNIER, Amsterdam; Dr. J. D. A. WAIBOER, The Hague; and Dr. J. ZELDENRUST, Leiden.

LITERATURE.

ALVING, TH., Flusspferdgeburt und andere Ereignisse im Zoo Kopenhagen. Zool. Gart., n.s., vol. V, pp. 34—37 (1932).

BENNEJEANT, CH., Anomalies et variations dentaires chez les Primates. Clermont–Ferrand, 258 pp., 223 figs. (1936).

BREUER, R., Über das Vorkommen sogenannter keilförmiger Defekte an den Zähnen von Ursus spelaeus und deren Bedeutung für die Paläobiologie. Palaeobiologica, vol. V, pp. 103—114, pl. VI—X (1933).

COLYER, F., Variations and diseases of the teeth of animals. London, VIII+ 750 pp., 1007 figs. (1936).

FLOWER, S. S., Contributions to our knowledge of the duration of life in vertebrate animals. Proc. Zool. Soc. London, pp. 145—234 (1931).

JENTINK, F. A., Catalogue Ostéologique des Mammifères. Muséum d'Histoire Naturelle des Pays-Bas, vol. IX, Leiden, 360 pp., 12 pls. (1887).

KERBERT, C., Over dracht, geboorte, puberteit en levensduur van Hippopotamus amphibius L. Bijdragen tot de dierkunde, Feestnummer Dr. Max Weber, pp. 185—191 (1922).

KUBACSKA, A., Pathologische Untersuchungen an ungarländischen Versteinerungen, V. Kieferknochen-Erkrankungen und Anomalien der Zähne bei dem Höhlenbären. Math. u. Naturw. Anz. d. Ungar. Akad. d. Wiss., vol. LII, pp. 695—712, pl. IV, V (1935).

———— ——, Id., VIII. Pathologisch veränderte Metapodien und Phalangen plistozäner Raubtiere. Palaeobiologica, vol. VI, pp. 214—219, pl. XVI—XVIII (1938).

SIGEL, W. L., Das Nilpferd des Zoologischen Gartens zu Hamburg. Zool. Gart., vol. XXIV, pp. 10—21 (1883).

VEVERS, G. M., Some notes on the recent birth of a Hippopotamus (H. amphibius) in the Gardens. Proc. Zool. Soc. London, pp. 1097—1100, 2 textfigs. (1926).

Fig. 1.

Fig. 2.

Fig. 3.

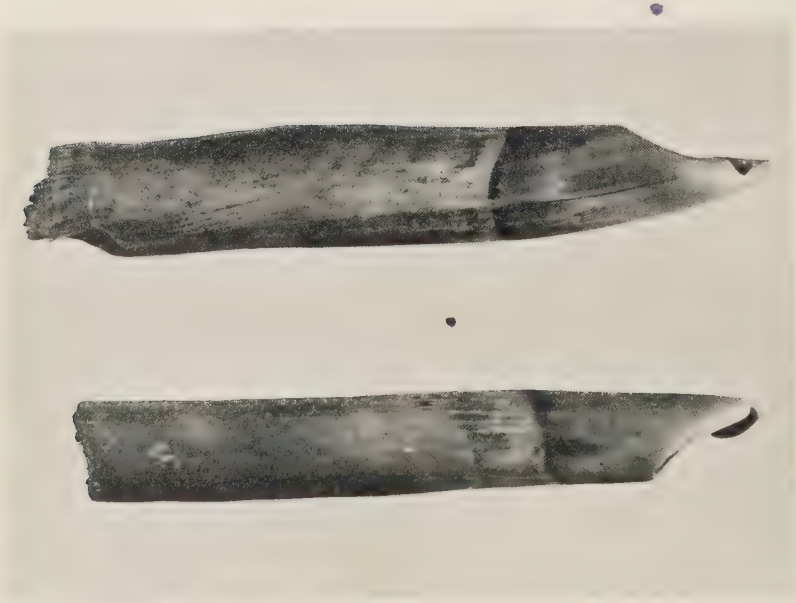


Fig. 4.

Fig. 5.

EXPLANATION OF FIGURES.

Fig. 1. *Hippopotamus amphibius* L. Rijksmuseum van Natuurlijke Historie, Leiden, cat. ost. n. Frontal part of the mandible showing the abnormal arrangement of the incisors. P_3 at the right side is rotated. The left I_2 has been removed. About $^1/_3$ nat. size.

Fig. 2. Vertical section through the left part of the symphysis, inner view. *a*: alveolus of the left I_2; *b*: alveolus of the left id_1; *c*: dental mass in situ; *d*: cancellated part of the bony structure; *e*: upper part of the symphysis with eburnification. About ½ nat. size.

Fig. 3. Vertical section through the right part of the symphysis, inner view. *a*: bony scale on the anterior wall of the alveolus of the right I_1. About ½ nat. size.

Fig. 4. Lower right and left I_2 from the specimen shown in fig. 1, viewed from above, showing the indentation in the left (upper fig.) and the grooves in the abraded surfaces. About ½ nat. size.

Fig. 5. Left frontal part of the maxillary, side view, showing the wedge-shaped defect on the posterior surface of I_1. About $^4/_5$ nat. size.

Comparative Physiology. — *Ueber den Tonus des Schneckenfusses (Helix pomatia L.).* I. *Der Tonus gemessen an der Pedalkonstanz.* (Vorläufige Mitteilung) [1]). Von N. POSTMA. (Aus dem Institut für vergleichende Physiologie der Universität zu Utrecht.) (Communicated by Prof. H. J. JORDAN.)

(Communicated at the meeting of October 25, 1941.)

A. Einleitung.

Es gehört zu den Funktionen des Gastropodenfusses (Aplysia, Helix, Arion, Limax) sich durch Zug an seinen Enden dehnen, d.h. plastisch deformieren zu lassen. Hierdurch reguliert er (übrigens auch der Rest des Hautmuskelschlauches) den Innendruck des Schizozöls. JORDAN hat diese Funktion früher mit der auxotonischen Methode untersucht, da diese den biologischen Verhältnissen der Selbstentlastung bei Dehnung entspricht. Hierbei kommt es zu einem Grenzwert der Belastung, mit der die Viskosität (der sogenannte Tonus) dieser glatten Muskeln im Gleichgewicht ist, sodass weitere plastische Dehnung vom Erreichen dieses Zustandes an unterbleibt. Diesen Zustand nannte JORDAN (5 und 6) die „*Konstanz*". Je nach den Versuchsbedingungen werden wir verschiedene Formen der Konstanz wie folgt unterscheiden:

I. **Pedalkonstanz.** Die Ganglien, vor allem die Pedalganglien, sind vorhanden.

a. *Muskelkonstanz* nennt JORDAN in diesem Falle das in Frage stehende Gleichgewicht bei niederer Belastung.

b. *Zentralkonstanz*, die besonderen Gesetzen entspricht, tritt bei hoher Belastung auf.

II. Wenn man alle Ganglien vor der Belastung exstirpiert, bleibt der Muskulatur als Zentralnervensystem das immanente Nervennetz (BIEDERMANN). Obwohl der Einfluss dieses Netzes auf den Tonus noch unbekannt ist, nennen wir das entsprechende Gleichgewicht in diesem Zustande, um der Möglichkeit des Bestehens einer solchen niedrigsten zentralen Innervation des Tonus gerecht zu werden, **„Netzkonstanz"**.

Die verschiedenen Formen von Konstanz sind offenbar Masse für den Grad des Tonus. Man wird sich fragen müssen, an welcher Stelle der Dehnungskurve (Zeit-Längenkurve) die Konstanz eintritt, also bei welcher Länge, nach welcher Dehnungszeit und, vor allem bei auxotonischer Methode, bei welcher Last. Je niedriger der Tonus ist, desto geringer wird die konstant getragene Last und desto länger der Muskel bei Eintritt des Gleichgewichtes sein. Weitere Fragen sind: wie ändert sich die Konstanz, wie steigt oder fällt der Tonus, wenn die Versuchsbedingungen geändert werden?

Ein zweites Mass für den Tonus ist der Widerstand des Muskels, der bei der Dehnung überwunden werden muss. Wie passt sich der sich dehnende Muskel der Last an, welche ist die Form der Dehnungskurve, deutet sie auf mehr oder weniger Widerstand gegen passive Dehnung? Bisher war in den Versuchen unseres Institutes dieser Widerstand, also der Neigungswinkel der Dehnungskurve gegen die X-Achse, das am meisten gebrauchte Mass für den Tonus. Aufgabe dieser Arbeit ist unter anderem, in dieser Beziehung auch die verschiedenen Formen der Konstanz zu ihrem Rechte kommen zu lassen; wir beschränken uns in dieser Mitteilung auf die Pedalkonstanz.

JORDAN hat die nachfolgenden Unterschiede zwischen diesen Konstanztypen feststellen können:

Muskelkonstanz (mit G. pedale, bei geringer Last) ist niedriger (d.h. sie tritt bei grösserer Länge des Fusses auf) als Netzkonstanz (ohne G. pedale) und die auxotonische Dehnungskurve des ganglienlosen Fusses liegt bei niederer Last immer oberhalb der Kurve des Muskels, der mit dem Pedalganglion in Verbindung steht.

Zentralkonstanz (mit G. pedale, bei hoher Last) ist höher als Netzkonstanz, aber in der Dehnungskurve tritt eine zwiefältige Ganglienwirkung auf: erst Beschleunigung der Dehnung, bis zu einem gewissen Punkte, dann aber Umkehrung dieser tonushemmenden

---

[1]) Mit Unterstützung der Niederl. Regierung aus den Einkünften der „Zomerpostzegels".

Funktion, so dass zuletzt das ganglienlose Tier stets tiefere Konstanz zeigt als das normale. Diese Umkehrung der Ganglienwirkung hat JORDAN (14) als *„Umschlag"* bezeichnet. Da die auxotonische Dehnungskurve, die zu Netzkonstanz führt, schliesslich bei Eintreten der Zentralkonstanz diejenige des Objektes mit G. pedale überholt, schneiden beide Kurven einander.

Durch folgende Methoden kann man die Konstanz beeinflüssen:

1. *Chemisch: a.* Bepinselung der Pedalganglien mit 2% Kokainlösung dämpft deren Aktivitätszustand durch schwache Kokainisierung derart, dass sie Erniedrigung der Pedalkonstanz bedingt. Steigerung des Erregungszustandes durch Bepinselung der genannten Ganglien mit Kochsalzlösung oder durch Auflegung von Kochsalzkristallen hat Erhöhung der Pedalkonstanz zur Folge. *b.* Injektion einer schwachen Dosis Kokain in die Muskulatur oder Bepinselung des Fusses mit diesem Alkaloid setzt den Tonus herab und verursacht Abschwächung der Muskelkonstanz.

2. *Reflektorisch:* In JORDANs „Halbtierversuchen" wird der Schneckenfuss durch einen Schnitt in der Medianebene derart in seine beiden symmetrischen Hälften geteilt, dass diese nur noch durch das Zentralnervensystem verbunden sind. Ein analoges Präparat gewinnt JORDAN nach Exstirpation der Ganglien, dadurch, dass er bei der Halbierung des Fusses eine rostrale Muskelbrücke ausspart. Beide Hälften sind nunmehr lediglich durch das Nervennetz in dieser Brücke mit einander verbunden.

Die eine Fusshälfte des Präparates mit Ganglien wird durch mittelhohe Belastung zur Konstanz gebracht. Wir nennen diese Hälfte die „registrierende" Hälfte; sie befindet sich nunmehr offenbar in einem Zustand, den man „Mittelkonstanz" nennen könnte [1]). Wird nun die andere Hälfte mit hohem Gewicht belastet, dann schwindet die Konstanz der „registrierenden Hälfte": sie lässt sich aufs Neue durch unveränderte Belastung dehnen; Entlastung der schwer belasteten Hälfte hat Wiederverkürzung der registrierenden zur Folge. Es tritt daher in beiden Hälften eine Koordination des tonischen Muskelzustandes auf.

Eine derartige Koordination zeigt sich nicht nur, wenn die Tonusänderung durch Belastung oder Entlastung hervorgerufen wird; man kann den Tonus der einen Hälfte auch durch Reizung steigeren, oder durch Kokaïnisierung vermindern, immer folgt ihr die registrierende Hälfte.

Im wesentlichen die gleichen Erscheinungen erhält man, wenn die beiden Hälften nur durch eine Muskelbrücke miteinander verbunden sind und die Ganglien fehlen. Jedoch sind dann die Tonusänderungen in der registrierenden Hälfte u.a. wegen des starken Dekrements und der geringeren Leitungsgeschwindigkeit in den Nervennetzen, viel schwächer; sie äusseren sich am besten durch entsprechende Aenderungen der Steilheit der Dehnungskurve. Ein ähnlicher aber geringerer Unterschied als zwischen Pedal- und Netzkonstanz ergibt sich für Zentral- und Muskelkonstanz: Bei dieser fand JORDAN im Halbtierversuch bei niedrigerer Belastung „unter sonst gleichen Bedingungen in sehr geringem Grade Tonusfall, und Entlastung bedingt Stillstand, keine Steigerung" (5, S. 214 u. 6, S. 547).

Von 1907 an ist die auxotonische durch die isotonische Dehnungsmethode ersetzt worden. Der ganglienlose Fuss lässt sich bei dieser konstanten Belastung nach einer kleinen, elastischen Streckung langsam und gleichmässig dehnen (11 u. 12). Ueber eine Konstanz wird in den entsprechenden Arbeiten JORDANs nicht mehr berichtet. Da die Muskellänge an sich kein Mass für den Tonus ist, wird die Dehnungsgeschwindigkeit, das heisst die Längenänderung als Funktion der Zeit, als solches verwendet. Es ist noch zu bemerken, dass nach 1907 ein Tragen der Last durch die Muskulatur ohne Ganglien an Kontraktur, einem pathologischen Zustande, zugeschrieben und also bei der Beschreibung und Deutung der Tonuserscheinungen des Schneckenfusses ausser Acht gelassen wurde. Daher hat JORDAN keine Netz- neben der Pedalkonstanz unterschieden.

[1]) Später handelt es sich um „die Reaktion einer Fusshälfte auf eine geringe Last", also um Muskelkonstanz (7, S. 215 gegenüber 5, S. 213).

Andererseits werden von 1907 an die Muskel- und Zentralkonstanz unter dem Namen Zentraltonus zusammengefasst. Sowohl bei geringer wie bei hoher Belastung zeigt der unverändert belasteten Muskel mit Pedalganglien nach einem Umschlag (Abb. 1 bei A′) ein Tragen der Last und die Dehnungskurve kreuzt immer jene des ganglienlosen Fusses (Abb. 1 in B). Für den Zentraltonus ist die Länge des Muskels bei dieser Konstanz das Mass für den Tonus.

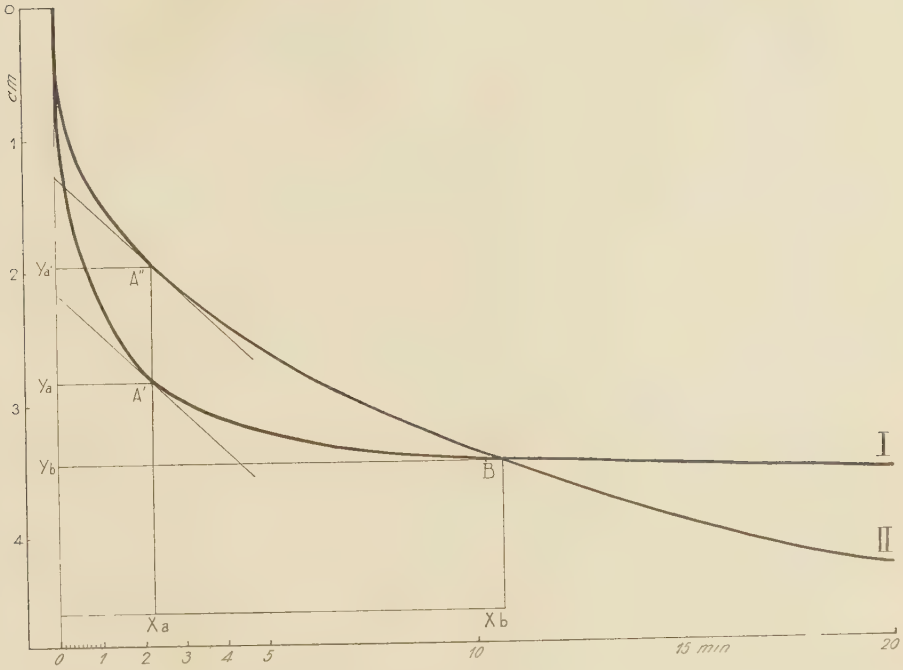


Abb. 1. Dehnungskurven zweier Helixfüsse: I mit Pedalganglien, II ohne Ganglien. Der Fuss mit Ganglien zeigt zuerst wesentlich niedrigeren Tonus als der andere (Tonuslösung durch die Pedalganglien). In A′ erfolgt durch Ganglieneinfluss der „Umschlag": Der Neigungswinkel der Tangente ist gleich demjenigen von Kurve I in Punkt A″ (gleiche Dehnungsgeschwindigkeit). Der Umschlag tritt gut 2 min nach Belastungsanfang auf; durch ihn wird daher der „Tonusspiegel" in beiden Füssen für kurze Zeit gleich gemacht. In B, in dem beide Kurven einander schneiden (fast 11 min nach Belastungsanfang), ist das Tonusniveau beider Füsse für ganz kurze Zeit gleich, bei verschiedenem Widerstand gegen Dehnung.

Bei meinen Versuchen über die Wiederherstellung des Tonus nach seinem Verlust durch passive Dehnung hat sich aber gezeigt: **1.** Dass der Helixfuss mit Pedalganglien ganz verschieden reagiert je nach dem die Belastung hoch oder niedrig ist, dass sich also auch bei der Dehnung mit konstanter Last Unterschiede ergeben, die einer Zentral- und einer Muskelkonstanz entsprechen. **2.** Weiter ergaben sich Umstände, unter denen die Länge, bei welcher der Pedaltonus die Last trägt, d.h. der Zustand, den ich Pedaltonus*niveau* nennen werde (siehe unten), sich verändert. **3.** Endlich zeigte sich, dass bei dem seiner Ganglien beraubten Helixfuss auch Konstanz auftreten kann, trotzdem in meinen Versuchen konstante Last zur Anwendung kam und die Objekte sicherlich normal waren.

Unter Umständen kann diese Netzkonstanz auf dem gleichen Niveau liegen wie die Pedalkonstanz. Hierüber wird in einer folgenden Mitteilung berichtet werden.

Bevor ich näher auf meine Ergebnisse eingehe, noch eine Bemerkung zur Nomenklatur: Wenn man mit „Tonus" das eine Mal die Länge des Muskels, das andere Mal den Dehnungswiderstand andeutet (5, S. 203; 6. S. 547/48), so gibt es nur einen einzigen Zustand, bei dem beide Definitionen gelten, nämlich während absoluter Konstanz, bei idealem Zentraltonus. Unter allen anderen Bedingungen müssen wir schärfer definieren um Misverständnisse zu vermeiden. Ich habe daher (23) unterschieden:

Tonusniveau, als die *Länge* des Muskels, unabhängig von dem gebotenen Widerstand;

Tonusspiegel, als der *Dehnungswiderstand* bei beliebiger Länge und Anwendung gleicher Last in allen verglichenen Versuchen.

B. Methodisches.

Wo liegt das Zentraltonusniveau? Im Falle eines idealen Zentraltonus (Kurve I, Abb. 1) ist die Antwort leicht zu geben: man misst die Länge des Muskels in Punkt B, und die Anfangslänge (O) und addiert hierzu die Dehnungsstrecke O—Yb.

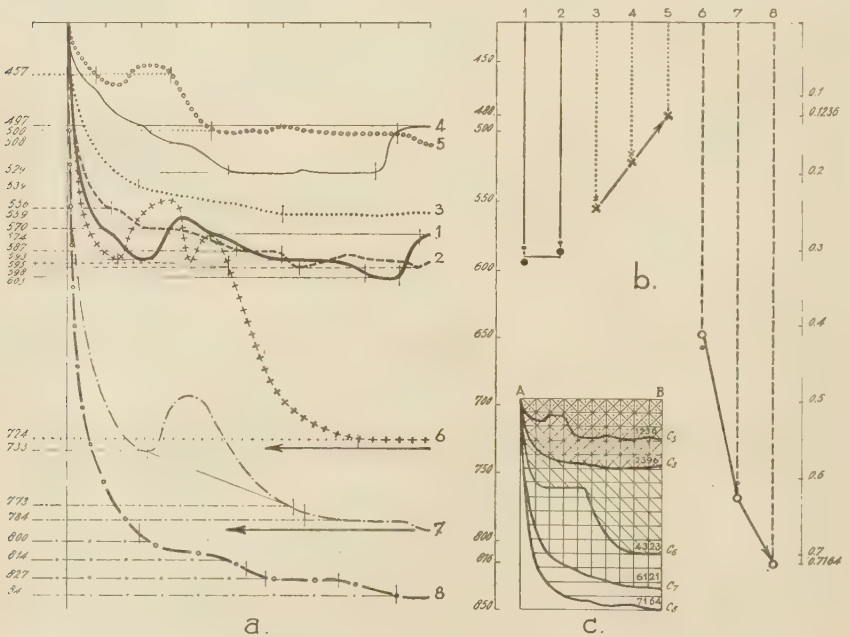


Abb. 2. Erklärung im Text.

Schwieriger aber wird die Beantwortung unserer Frage, wenn zwar Konstanz eintritt, diese aber nach kürzerer oder längerer Zeit mit einer anderen bei geringerer oder grösserer Länge vertauscht wird. (Abb. 2 a, Kurve 5, 4, 2, 6 u. 8). Ich habe dann ein mittleres Niveau berechnet aus den verschiedenen gemessenen Niveaux, der Dauer ihrer Aufrechterhaltung und zwar bei einer auf 10 min beschränkten Dauer der Dehnung; Mittleres Niveau $= \frac{t_1 . l_1 + t_2 . l_2 + \ldots + t_n . l_n}{t_1 + t_2 + \ldots + t_n}$, z.B. für Kurve 5: $\frac{2 \times 4{,}57 + 5 \times 5{,}00}{2 + 5} = 4{,}88$ cm.

Zur Kontrolle habe ich die Oberflächen zwischen den Kurven 5 bis 8 und das Achsensystem AB—BC_5/BC_8 (Abb. 2 c) ausgeschnitten und die Papierstücke gewogen (dritte Kolumne der Tabelle):

	Berechnet	Gewogen
Kurve 5	4.88 cm	0.1236 g
Kurve 3	5.54 „	0.2396 „
Kurve 5	6.47 „	0.4323 „
Kurve 7	7.66 „	0.6121 „
Kurve 8	8.19 „	0.7164 „
Differenz zwischen höchstem und niedrigstem Niveau (8—5)	3.28 cm	0.5928 g

In der Abb. 2 b sind die gefundenen Werte dergestalt als Ordinaten aufgetragen (links die Längen in 0,01 cm, rechts die Gewichte), dass die Extreme (Niveaux 5 und 8) je auf einem Punkte der Y-Achse zusammenfallen: Länge 4,88 = Gewicht 0,1236; Länge 8,19 = Gewicht 0,7164. Es zeigt sich dann, dass die Partner der übrigen Wahrnehmungspaare (Längenmerkmale dick, Gewichtszeichen dünn gezeichnet) fast aufeinanderfallen. Die Methode der Ausmessung der Kurven und Berechnung der Niveaux ist also befriedigend.

C. Ergebnisse.

1. *Einfluss der Belastung.*

Es gibt ziemlich viele Fällen (ungefähr 40%), in denen jede Ueberkreuzung der Dehnungskurven zweier Füsse, von denen bei dem einen die übergeordneten Ganglien entfernt wurden, ausbleibt. Die Kurve des Fusses ohne Ganglien läuft dann dauernd oberhalb oder unterhalb der pedaltonischen Kurve (Dehnungskurve des Fusses mit Pedalganglien). So drängt sich die Frage auf, ob das Niveau der Pedalkonstanz und der Tonusspiegel des ganglienlosen Fusses oder, seltener, das Niveau der Netzkonstanz von äusseren Faktoren bestimmt sein können, Faktoren, die noch aufzusuchen wären, oder von unkontrollierbarer endogener Natur sind.

Ergebnisse des Studiums der pedaltonischen Wiederhohlungskurven an ein und demselben Fusse (17, 23, 24 und 25) deuteten darauf hin, dass die Länge, welche bei einer vorhergehenden Dehnung erreicht wurde, grossen Einfluss auf das Niveau der Pedalkonstanz der nächstfolgenden Dehnung habe: Kurve 7, Abb. 2 a, zeigt den Umschlag auf dem Niveau, auf welchem die Kurve 6 endete, Kurve 8 desgleichen in Bezug auf 7 (man beachte die Pfeile). Nun wird der Schneckenfuss vor dem Dehnungsversuche, nämlich bei der Präparation (vgl. 11 u. 12), immer etwas ausgedehnt und somit mehr oder weniger gespannt. Das Präparat erreicht dadurch eine bestimmte Länge, die wir die **„Spannungslänge"** nennen wollen. Tatsächlich stimmt die Länge des Fusses, die nach der eigentlichen Dehnung, durch die Pedalkonstanz behauptet wird, mit der Spannungslänge überein:

Versuchsserie PK-R-L . C, *7. u. 8.-III-1934. 12 Schnecken von 18 g; Spannungslänge* 6¼ *cm; Belastung variiert.*

Belastung (jede Gruppe 4 Tiere)	5 g	10 g	20 g
Niveau in cm Länge	**6.13** ± 0.37	6.32 ± 0.61	**6.27** ± 0.71
Dehnungsstrecke in cm	1.40 ± 0.38	1.67 ± 0.53	2.12 ± 0.37

Versuchsserie PK-R-L/G, *19. u. 20.-III-1934. 11 Schnecken von 18 g und 4 von 21 g; Belastung 20 g; Spannungslänge 6 oder 7 cm.*

Grösse der Tiere	18 g (6 Füsse)	18 g (5 Füsse)	21 g (4 Füsse)
Spannungslänge	**6** cm	**7** cm	**7** cm
Niveau in cm Länge	**5.97** ± 0.32	**6.97** ± 0.38	**6.83** ± 0.32

Die Grösse der Last und das Verhältnis zwischen Last und Grösse des Objektes haben keine Verschiebung des Niveaus zurfolge; *nur die Spannungslänge ist bestimmend.*

Dass die Grösse der Last gar keinen Einfluss auf das Niveau hat, gilt nicht für den Zentraltonus bei Wiederholungskurven an ein und demselben Fusse. Es ist uns gelungen (vgl. 13, 15, 16, 17 und 23) durch ein besonderes Verfahren nach tonischer Dehnung den ursprünglichen Zustand derart wieder herzustellen, dass man die Dehnungskurve des ruhenden Fusses von der anfänglichen Länge aus beliebig oft wiederholen kann (Abb. 2, Kurven 1 und 2). Um solche identische Wiederholungskurven zu erhalten, ist es erforderlich den Helixfuss mit mittelhoher Last zu dehnen und darauf ganz Widerstandslos, wenn nötig mittels schwacher „reflektorischer" Reizung, zur Verkürzung zu bringen.

Welche Belastung als „mittelhoch" gilt, muss jedesmal experimentell bestimmt werden. Im Beispiel der Abb. 2 war es 10 g; geringere Last (5 g) bringt das Niveau der aufeinanderfolgenden Kurven immer höher: Kurve 3, 4 und 5. Grössere Last (20 g) setzt das Niveau stets mehr herab (Kurven 6, 7 und 8). In Abb. 2 b sind die Abänderungen der Niveaux schematisch dargestellt.

Ich habe einige Anweisungen dafür, dass die Mittelbelastung bestimmt ist durch die Last, welche der Fuss vor der Präparation zu tragen hatte. Für mittelgrosse Schnecken (18—23 g Gewicht) ist die Mittelbelastung etwa 10 g.

In Bezug auf das Niveau der pedaltonischen Wiederholungskurve ist also nur eine bestimmte Belastung indifferent, kleinere und grössere Lasten ändern das Niveau in bestimmter Weise. Das Niveau der ersten Dehnungskurve zeigt diese Wechselbeziehung nicht.

2. *Einfluss der Kälte und Wärme.*

Der Tonus des Schneckenfusses verhält sich in seiner stationären Form der Temperatur gegenüber wie ein plastischer Körper: die Kälte steigert den viskösen Widerstand des seiner Ganglien beraubten Fusses, Wärme vermindert ihn.

Auch die Pedalkonstanz (Zentraltonus) hat JORDAN einer Viskositätserhöhung zugeschrieben (8), die ebenfalls durch Wärme aufgehoben werden kann (9 und 10). Wenn die Konstanz lediglich auf Viskosität zurückzuführen wäre, so müsste nicht nur Wärme das Niveau verminderen, sondern Kälte müsste es steigern. Dieses letztere ist aber nicht der Fall, wie sich aus einer Reihe von Versuchen ergab.

Kälte hat nämlich keinen Einfluss auf das Niveau, Wärme über 22° C. setzt dahingegen in der Tat das Niveau herab: bei 27° ist es um 11.6 %, bei 32° um 19.8 % erniedrigt. Zwei Ursachen könnte man für diese Verminderung verantwortlich machen. Neben Herabsetzung von Viskosität durch Wärme, käme vielleicht auch eine Wirkung der Wärme auf das zentrale Nervensystem in Frage. Wir werden dies späterhin zu untersuchen haben.

3. *Einfluss vom Wassergehalt des Fusses.*

Ich habe (26, 27, 28) die Bedeutung der Hydratation des Fusses für den peripheren Tonus feststellen können:

Während des Winterschlafes sind die Schnecken wasserarm; der Wassergehalt des Fusses weist nach BELLION (1) und FISCHER (3) 9 % Verlust auf. Ich habe Füsse solcher wasserarmen Tiere gedehnt und die Kurven verglichen mit entsprechenden Kurven von Präparaten, die Tieren entnommen waren, die reichlich Wasser aufnehmen konnten. Dieser Wasseraufnahme belief sich, bezogen auf das Gewicht der trocken gehaltenen Schnecken nach Abzug des Gehäuses auf etwa 37 %; VON BRAND (2) fand 21 %, HOWES & WELLS (4) 30 % Gewichtszunahme. Der Wassergehalt der Fussmuskeln wurde nach Aufnahme der Dehnungskurve bestimmt; sie zeigen eine durchschnittliche Zunahme von nur 4 %. Gegenüber dieser geringen Aenderung stand ein sehr beträchtlicher Unterschied im tonischen Widerstand: 10 min nach der Belastung war die Dehnungsstrecke der feucht gehaltenen Tiere um 43 %, der Elevationswinkel um 38 % grösser als bei den trocken gehaltenen Vergleichsexemplaren. Das aufgenommene Wasser, wovon nur etwa ein Zehntel dem Fusse zugute kommt, vermindert also den viskösen Widerstand sehr

beträchtlich. Ueberdies verursacht diese Fraktion einen auffallend gleichförmigen Verlauf der Dehnungskurven von Füssen ohne Ganglien, während bei den trockenen Tieren zwischen den Kurven verschiedener Füsse grosse Unterschiede vorkommen (26, Abb. 1). Trockenheit kann daher offenbar in verschiedenen Graden auftreten, im Gegensatz zur Sättigung mit Wasser. Erniedrigt dieses Wasser auch das Niveau der Pedalkonstanz?

Wir vergleichen das Niveau des Pedaltonus von wasserreichen und wasserarmen Schneckenfüssen mit demjenigen von Füssen mit normalem Wassergehalt. *Extreme Wasseraufnahme beeinflusst das Niveau der Pedalkonstanz nicht* (im Gegensatz zum ganglienlosen Fuss, bei dem der visköse Widerstand durch Wasseraufnahme stark abnimmt). *Wasserarmut dahingegen erhöht das Niveau dieser Konstanz.* In drei Versuchsserien zu 13, 14 und 12 Tieren erhielt ich folgende Mittelwerte dieser Steigerung in % der Spannungslänge: Das Niveau steigt um 17.5, 13.0 und 11.3 %. Hierbei betrug der Wasserverlust im Mittel 2.5 % und war der Durchschnitt aller Niveausteigerungen bei 39 Tieren 12.5 %.

4. *Einfluss des jeweiligen Aktivitätszustandes der Pedalganglien auf das Niveau der Pedalkonstanz.*

JORDAN und seine Mitarbeiter (5, 6 u. 16) haben gezeigt, dass der Antivitätszustand der Pedalganglien das Niveau der Pedalkonstanz entschieden beeinflusst (Versuche bei denen Kokain oder Kochsalz auf die Ganglia pedalia gebracht wird (siehe S. 1152)). Wenn wir annehmen, dass auch normalerweise Schwankungen des Aktivitätszustandes der Tonuszentren auftreten, so sind wir imstande zu erklären warum in etwa 40 % der Fälle die Dehnungskurve eines Fusses mit Ganglien und diejenige eines Fusses ohne Ganglien einander nicht schneiden: Falls nämlich der Fuss mit Ganglien sich unter Einfluss seines Zentrums in niederem tonischen Aktivitätszustand befindet, kann das Niveau der Pedalkonstanz so niedrig zu liegen kommen, dass die Netzkonstanz dauernd oberhalb dieses Niveau bleibt. In solchen Fällen ist der tonus*lösende* Einfluss der Ganglien auf die Dehnungskurve (vor dem Umschlage) besonders gross, während eine solche Hemmung bei der Dehnung des ganglienlosen Fusses naturgemäss fehlt.

Eine entgegengesetzte Kombination trifft man bei einem Vergleichspaar, bei welchem der Fuss mit Ganglien auf eine hohe „Mittellast" eingestellt ist, sodass die benutzte geringere Belastung die Senkungsgeschwindigkeit der ersten steilen Strecke nicht erhöht (vgl. Mitt. II: Die Tonuslösung). Hat dieser Fuss überdies einen hohen Tonusspiegel und der ganglienlose einen niedrigeren, so kann die pedaltonische Kurve bis zum Umschlag neben oder ganz oberhalb der Kurve des Fusses ohne Ganglien bleiben. Auch dann werden solche Kurvenpaare einander nicht schneiden. Wir kommen in einer gesonderten Mitteilung auf solche „tonische Stimmungen" unserer Objekte zurück (Vergl. auch Lit. No. 19 und 20).

Tatsächlich wird das Niveau der Pedalkonstanz durch schwache Kokainisierung der Pedalganglien herabgedrückt und zwar um durchschnittlich 13,8 % der Spannungslänge. Kochsalzlösung, die man auf gleiche Weise auf die Pedalganglien aufträgt, steigert dahingegen das Niveau. In zwei Versuchsserien erhielten wir als Mittelwerte von je 8 Versuchen eine Steigerung von 12.4 und 19.2 % der Spannungslänge.

D. Deutung der Ergebnisse und Zusammenfassung.

JORDAN hat am ganglienlosen Helixfuss vor allem die Statik des Tonus untersucht, um für sie im Verhalten von plastischen Stoffen ein Modell zu finden. Unsere Aufgabe war, weiter zu gehen auf dem von uns schon früher eingeschlagenen Wege, die Dynamik des ganglienlosen Fusses zu studieren, durch welche er sich von den nicht biologischen Modellen und dem Fuss mit Ganglien unterscheidet, und daher ausgesprochener Dynamik, näher kommt. Hierzu musste auch die Pedalkonstanz aufs Neue untersucht werden. Die Einflüsse, die diese Konstanz veränderen, wie Kokain, Kochsalz (5 und 6), Belastung (9 und 10), hatte man früher erst nach Eintritt der Pedalkonstanz einwirken lassen; oder aber der betreffende Eingriff hatte zwischen der Aufnahme von Wiederholungskurven stattgefunden (Last 23; Kokain, Kochsalz 16). Nunmehr wurden die betreffenden Ver-

suchsbedingungen schon vor Applikation der ersten Belastung verwirklicht und sodann die reaktive Veränderung der Pedalkonstanz quantitativ mit Normalkurven verglichen. Dies war möglich geworden, da wir in der Spannungslänge ein Normalmass für das Konstanzniveau hatten. Untersucht wurde ausserdem, welchen Einfluss Wärme, Kälte und Wassergehalt des Fusses auf die Lage des Pedalkonstanzniveaus haben.

Wärme und zwar erst oberhalb 22° C vermindert das Niveau der Pedalkonstanz. Dehydratation steigert es: weder Kälte noch reichliche Wasseraufnahme ändern das Niveau. Diese Einseitigkeit der Wirkung stützt nicht die Meinung, dass Konstanz lediglich durch gesteigerte Viskosität zuwegegebracht wird. Bei vergleichenden Versuchsserien braucht man daher nicht genau auf den Wassergehalt der Versuchstiere und auf die Temperatur während der Dehnung zu achten, solange man den extremen Grad dieser beiden Faktoren vermeidet.

Biologisch sind diese Verhältnisse wohl zu verstehen: bei extremer Wärme und bei Trockenheit kriechen die Schnecken nicht. Sie sind dann meist zurückgezogen in ihrem Gehäuse. Sie brauchen also keine Regulierung des Muskelzustandes während des Kriechens bei sehr warmer, trockener Witterung. Offenbar wird dahingegen der Einfluss der Kälte und ausgiebiger Wasseraufnahme bei Regen irgendwie ausgeglichen: Diese beiden Faktoren stören hierdurch die Leistungen der Muskulatur offenbar nicht. Für die Hydratation mag das damit zusammenhängen, dass die Muskelkolloide über einen gewissen Sättigungsgrad kein Wasser aufnehmen.

LITERATUR.

1. BELLION, M., Recherches expérimentales sur l'hibernation de l'escargot (Helix pomatia L.). Diss. Lyon, No. d'ordre 44, 1909 und Ann. Un. Lyon (N. S.), **27**, 1909; *2.* BRAND, TH. v., Zs. vergl. Physiol. **14**, 200, 1931; *3.* FISCHER, P. M., Recherches sur la vie ralentie de l'Escargot (Helix pomatia L.). Thèse sci. natur., Paris, 1931; *4.* HOWES N. H. und G. P. WELLS, J. exp. Biol. **11**, 328, 1934; *5.* JORDAN, H. J., Pflüg. Arch. **106**, 189. 1905; *6.* JORDAN, H. J., Ibid. **110**, 533, 1905; *7.* JORDAN, H. J., Ergebn. Physiol. **16**, 87. 1918; *8.* JORDAN, H. J., Biol. Zlbl. **39**, 462, 1919; *9.* JORDAN, H. J., Verh. D. Zool. Ges. **31** Jahresvers. Kiel 108, 1926; *10.* JORDAN, H. J., und J. D. F. HARDENBERG, Zs. vergl. Physiol. **4**, 545, 1926; *11.* JORDAN H. J. und G. C. HIRSCH, Uebungen aus der vergleichende Physiologie, Springer, Berlin, S. 224 f.f. und 241 f.f., 1927; *12.* JORDAN, H. J. und P. J. v. D. FEEN, Abderhalden's Hb. Biol. Arb. Meth. Abt. IX, Tl. 4, S. 295 f.f. und 357 f.f., 1929; *13.* JORDAN, H. J., Arch. Néerl. Zool. **1**, 1, 1934; *14.* JORDAN, H. J., Proc. Kon. Akad. v. Wetensch., Amsterdam, **38**, 358 (1935); *15.* JORDAN, H. J., Ergebn. Physiol. **40**, 439, 1938; *16.* JORDAN, H. J., Frl. C. E. SLUITER und N. POSTMA, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 1032 (1938); *17.* MAAS, J. A., Arch. Néerl. Physiol., T. XXIII, 1, 1938; *18.* POSTMA, N., Proc. Kon. Akad. v. Wetensch., Amsterdam, **36**, 360 (1933); *19.* POSTMA, N., Acta Brev. Neerl. **4**, 106, 1934; *20.* POSTMA, N., Ned. Tschr. v. Geneesk., **78**, 3638, 1934; *21.* POSTMA, N., Acta Brev. Neerl., **4**, 109, 1934; *22.* POSTMA, N., Ned. Tschr. v. Geneesk., **78**, 3640, 1934; *23.* POSTMA, N., Onderzoekingen betreffende het herstel van de tonus bij de slakkevoet (Helix pomatia L.) door prikkeling en betreffende de invloed van het centrale zenuwstelsel daarop. Diss. Utrecht 1935; *24.* POSTMA, N., Acta Brev. Neerl., **5**, 149, 1935; *25.* POSTMA, N., Proc. Kon. Akad. v. Wetensch., Amsterdam, **38**, 1036 (1935); *26.* POSTMA, N., Ibid., **39**, 891 (1936); *27.* POSTMA, N., Acta Brev. Neerl., **6**, 119, 1936; *28.* POSTMA, N., Ned. Tschr. v. Geneesk., **80**, 3629, 1936; *29.* SWINDEREN, J. W. DE MAREES VAN, De regeling van den tonus in de spieren van Helix pomatia. Diss. Utrecht 1927; *30.* SWINDEREN, J. W. DE MAREES VAN, Arch. Néerl. Physiol., **13**, 571, 1928.

Medicine. — *Some considerations concerning our knowledge of the action of the auditory organ.* Part II. By J. BIJTEL. (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of October 25, 1941.)

Surveying the present state of our knowledge we can distinguish for the sake of the summary:

I. The "leading thoughts" forming the basis of the investigations.

II. The technical methods.

This distinction cannot always be sharply maintained.

An important starting-point of recent investigations and views about the activity of the hearing-apparatus, more especially about the process of conduction in the auditory nerve, is the idea that each cycle of a sound-stimulus gives one nerve-impulse. In the physiology of the auditory organ, more often than with regard to any other organ, the study of the process of conduction in the centripetal nervous elements has come to the fore. This train of thought can be understood very well. The mechanical phenomena attending sound can be determined very well at least at the source of sound, as regards form and size and when they are transmitted as an identical vibrational process through the auditory nerve, this also holds good for the attending phenomenon, which we then ought to perceive as an action-current of the nerve. This doet not hold good e.g. for a light-stimulus on the eye. It is true that the wave-length of light can be measured accurately and that the corresponding vibrational number can be determined, but there is no electric measuring-apparatus which might registrate action-currents of the frequency of light at the optical nerve directly. With regard to the sensation of taste, smell, pain and pressure, we are dealing with stimuli of quite a different character, not with a series of equal impulses working on the peripheral organ and coming back periodically. The sensation of vibration might again be compared with hearing. So in a certain respect the auditory nerve is *the* object for seeking, experimentally, the connection between outer stimulus and the process of nerve-conduction. In working it out, meanwhile, the classical conception that the process of nerve-conduction is qualitatively independent of that which happens at the end-organ and has the same typical characteristics, for all the nerves, has been given up at once. The whole plan of the investigations about the process of conduction in the auditory nerve presupposes an exceptional position of this nerve. It is taken as a matter of course that each cycle of a sound-vibration gives a nerve-impulse. We pointed out already that the electrophysiological phenomena, as they are observed with a special technical method, point in this direction. In our opinion, however, the applied technical method is not of such a character that we may speak of a crucial experiment. We know further that a certain number of cycles is necessary for a full, well-defined sound-impression. Up to 500 cycles this number is fairly constant, namely 8—10. Supposing this number to be 10 cycles a second, the minimal duration for a tone of 100 cycles a second is equal to 0,1 second, for a tone of 500 cycles a second it is equal to 0,02 second. So one can say that up to 500 cycles a second the necessary minimal duration of the acting sound is inversely proportional to the frequency. Above 500 cycles a second the minimal duration diminishes less strongly with the frequency and increases a little above 5000 cycles a second. There must be a minimal duration, however, for a well-definable perception. Sounds of less than this minimal duration give a so-called click-perception, the power to distinguish pitch being then strongly weakened. One cannot draw a certain conclusion from this about the significance of this phenomenon. Assuming, however, — for the present as an hypothesis — that one nerve-impulse is

necessary and sufficient for a complete perception, one is led to think that the minimal number of cycles necessary for a complete perception gives one nerve-impulse, in other words that for one nerve-impulse a series of some cycles is necessary. Nor can this be quite right, because a series of a shorter duration than that which is minimally necessary for the originating of a well-definable impression still gives a certain perception. We only want to elucidate that on account of the above-mentioned facts in general doubt of the axiom: one cycle = one nerve-impulse and with this doubt of the now predominating views about the process of conduction in the auditory nerve, is fully justified.

Indissolubly connected with the above-mentioned classical conception of the process of conduction of excitation in a nerve is the principle of independent excitability, which means that several influences can have a direct exciting action on the nerve-fibres with the same — invariable — consequence as adequate excitation of the end-organ. If we try to approach the question from this angle, we can only do so if we take the auditory nerve of man, this being able to inform us of the ultimate result, the perception. As a stimulus we must in the first place consider the electric current. If, there being excitation with alternating currents of different frequencies, the sound-perception that arises is always the same, this can at once be considered to speak against the now predominating conception. In literature some data of such experiments are to be found, which point to the fact that such a simple state of affairs does not occur. Rather is it concluded that in certain circumstances the perceived sound is determined by the very frequency of the exciting current. This then ought to be attributed to excitation of the cochlea-apparatus, perhaps also of the middle ear. It is assumed that electrostatic influences in the peripheral auditory organ play a part here. So, to make a pure experiment, it would be necessary to excite, as an object, an auditory nerve without a peripheral organ, so e.g. after extirpation of the labyrinth. It remains to be asked in how far in this case one is still dealing with a sufficiently active auditory nerve.

The stimulation with electric currents also brings technical difficulties with it. For one does not stimulate the auditory nerve directly, but brings the current to the bodily surface. The density of the current in the auditory nerve is unknown here. When one uses a direct current with one electrode close to the outer ear and one so-called indifferent electrode at a distance one can certainly stimulate the nervus octavus, as appears from the phenomenon of the galvanic nystagmus. One may assume that this also holds good for alternating currents of a relatively low frequency up to e.g. 50 cycles a second. When the frequency is higher, however, one undoubtedly gets more complicated situations with the dividing of the current. Influences of capacity must be strongly felt in this region of less conducting bony tissue that has a very complicated relation to the non-bony tissue.

The demonstrative force of stimulation of the auditory nerve when operations are resorted to must always be considered limited because of morbid qualities or inferiority [1]). The same holds good for tinnitus, when often, too, the influence of a still present peripheral apparatus must be taken into account.

All things considered we may not yet at once expect a conclusive result from investigations in this direction. Nevertheless we think that we must pay attention to them.

A second item about which an axiom obtains is the analytical power of the auditory organ. We know from daily observation that the ear possesses a power to distinguish qualitatively, several sorts of sounds and tones. Probably the idea has arisen from this that the auditory organ in a physical sense analyses every sound in its components and that higher centres again build sound-images from these components. For this conception,

---

[1]) VON BÉKÉSY, [BÉKÉSY, G. VON, Zeitschr. f. technische Physik **17**, 522 (1936)] mentions such an experiment, in which during an operation the auditory nerve was stimulated by alternating currents of different frequencies between 1.5 and 6000 Herz. 1.5—30 Herz give murmurs of an impulsive character. At 30 Herz the impulses fuse to a tone. From 100—6000 Herz the exciting currents are followed by an invariable sound-impression. VON BÉKÉSY does not give any further details.

however, we have no sufficient arguments. HELMHOLTZ [1]) devotes to this subject a chapter of "Die Lehre von den Tonempfindungen". From this it appears that the separate perception of overtones from a sound is only possible with the help of tricks, so that special attention is drawn to a certain overtone. It appears that one must know the components that one wants to perceive separately. Such a component in itself must be produced as a tone and its image retained in the memory, so that immediately afterwards one can find this component again in a sound. The argumentation of HELMHOLTZ mainly limits itself to showing that the ear does not perceive overtones that cannot be shown physically in the sound. It appears further that the overtones do not only determine the timbre, but also the subjective impression of pitch. This subjective element in determining the pitch of a sound sometimes leads to differences in judgment, even among persons with very talented and trained ears, of which HELMHOLTZ gives striking examples. Something similar we also find mentioned in HOOGERWERF [2]). The curves of a vowel that is sung give fundamental tones that are sometimes 1 or 2 octaves lower than the singer, comparing with a tuning-fork, meant. HOOGERWERF attributes this to the varying sensitivity of the auditory organ to varying pitches. Owing to this it may happen that an overtone in a sound, e.g. the human voice, predominates, the pitch being determined subjectively.

In this connection we draw attention to the fact that impulses of a periodical character, but in themselves not having the form of a sinus, can bring about a tone-perception. SCHOUTEN [3]) showed this. In a compound sound with several higher overtones, from which the fundamental tone has been eliminated, this fundamental tone can be perceived. This tone is then caused by the periodical impulses of the complexes of the harmonics. SCHOUTEN expresses this in this way, that the low tone, the so-called residue, appears to be correlated to the periodicity of the vibration, which arises when the high harmonics are joined. This fundamental tone is described as a strong, sharp sound and can even be perceived beside the fundamental tone, which may be added again. This residue-tone gives no beats with a frequency that differs little from it and so does not occur as a vibration having the form of a sinus and being physically present in the sound. So at any rate the auditory organ appears also to be sensitive to physical impulses of a character quite different from the vibrations that have the form of a sinus. SCHOUTEN thinks that the localisation-theory can be made to agree with this phenomenon.

Also as regards this datum it is difficult to judge the physiological importance at once. We think, however, that we may say that the phenomenon points in this direction: the auditory organ has not so much a sharp analytical power, but is rather prepared to make all physical periodicity into sound-impressions.

The opposite of the ordinary conception — analysis in the peripheral followed by synthesis in the central organ — would bring with it that certain psychical images are formed of sounds which up to a certain point can be analysed. If the ordinary conception were the right one, then it is indeed strange that the recognising of separate components is so difficult. It is true that neither for the other conception can we give conclusive evidence, but in our opinion the axiom of sound-analysis in the auditory organ as the primary element in the function must again be considered as a problem.

II. The technical methods.

We already pointed out several technical difficulties, namely as regards the currents of action and the direct excitation of the auditory nerve.

As long as the currents of action are not determined by direct measurement at the auditory nerve without amplifiers according to the requirements of classical electrophysiology, one cannot draw conclusions from this phenomenon.

---

1) HELMHOLTZ, H., Die Lehre der Tonempfindungen, 2e Auflage, 1e Abt. Vierter Abschnitt, blz. 84, Braunschweig 1865.

2) HOOGERWERF, S., Onderzoekingen met de snaarphonograaf, Diss. Leiden 1924.

3) SCHOUTEN, J. F., Philips Technisch tijdschrift **5**, 298 (1940).

No less attention ought to be paid to an exact view of the sound-stimulus. Several attempts have been made to measure this stimulus. These attempts were nearly always limited to determining the energy at the source of excitation. In literature we only found a few data about the energy which reaches the tympanic membrane, namely measurements of the resonance in the auditory canal. FLEMING [1]) published an investigation regarding the measurement of the sound-pressure near the tympanic membrane. VON BÉKÉSY [2]) also drew attention to the resonance in the auditory canal. It appears that for tones in the resonance-region of the auditory canal the sound-pressure at the tympanic membrane is considerably higher than would be the case at the place of the tympanic membrane with free propagation of the sound. VON BÉKÉSY found for 2500 cycles a second a value about 5 times greater at the tympanic membrane. In this case vibrating membranes were used as sound-producers. When one uses tuning-forks the field of sound is so complicated that a calculation has not yet appeared possible.

In the circumstances in which we must work it is never possible to speak of a free propagation, unless this is perhaps approached when there are low tones. In every walled-in space standing waves arise. With the help of sound-damping walls one can reduce this phenomenon, but as soon as a person on whom the experiment will be made is brought into a room a surface that re-echoes sound is introduced. We have paid attention to this problem, which has led to the construction of a room with a special form by Prof. ZWIKKER [3]), in which room in principle it would not be possible for a sound-ray to return and interfere with an approaching wave. The principle appeared to be right. Inevitable technical imperfections, meanwhile, again cause standing waves above 500 cycles a second.

Now we have not at all yet given a full treatment of the subject of the technical methods. We only touched a few points and remarked that no technical execution gives a conclusive solution. By saying so we do not underrate the value of modern technical methods for practical purposes. We think, however, that the often far-reaching conclusions from routine-investigations are not considered critically enough. One should be aware of the fact that it is really only the true value of energy or sound-pressure at the tympanic membrane that may be considered as a measure for the sound-stimulus.

With the above we think we have sufficiently accounted for our remarks made at the beginning. In the first place there ought to be a critical reconsideration of our fundamental notions. In principle a different vision will not be an entirely new vision. Vitalistic and materialistic ways of thought have alternated in history and have been expressed in many ways. Though nowadays one cannot speak of a conflict between these extremes, they are still to be recognised as different biological ways of thinking also as regards the subjects treated here. It cannot at all be said that a purely materialistic way of thinking predominates. Nevertheless, with regard to the auditory organ the identification with a physical instrument often comes too much to the fore. We shall always have to be aware of the fact that the auditory organ ultimately has a psychical function in the living organism. We must be aware of the fact that the function of the whole organ is directed to this and determined by it.

The realisation that the many special problems which have arisen in course of time are removing us further and further from the main point — a satisfactory notion of the function of hearing — led us to the above considerations and the indicating of lines of action along which our knowledge, which is now of a rather chaotic character, will have to come to a more correct conception.

---

[1]) FLEMING, N., Nature, **143**, 642 (1939).

[2]) BÉKÉSY, G. VON, Acta Otolaryngologica **27**, 281 (1939).

[3]) ZWIKKER, C. Nederlandsche Keel-neus-oorheelkundige Vereenigng, Amsterdam, Verslag der **64**e vergadering, 13 (1937). Ned. Tijdschr. v. Geneesk. **82**, 4281 (1938).

BIJTEL, J., Nederlandsche Keel-neus-oorheelkundige Vereeniging, Amsterdam, Verslag der **66**e vergadering, 9 (1938). Ned. Tijdschr. v. Geneesk. **83**, 3649 (1939).

ERRATA dans la communication de Prof. J. WOLFF: *„Inégalités remplies par les fonctions univalentes."* (Proc. 27 Sept. 1941):

p. 956 ligne 10 z_n *lire* z

p. 956 ligne 25 cependent *lire* cependant

p. 960 ligne 14 $><$ à supprimer

p. 960 ligne 22 indépendente *lire* indépendante

p. 960 ligne 27 *lire* tels que, z tendant vers *it* sur Γ_t,

p. 960 lignes 31, 32, 33, 34 *lire*

THÉORÈME VI A. À toute famille de courbes $\Gamma_\alpha \left\{ 1 - |z| = \psi\left(\arg \frac{z}{\alpha}\right)\right\}$ dans D, tangentes aux points α à la circonférence $|z| = 1$, $\psi(u)$ étant une fonction convexe, il correspond une plénitude de points α tels que, z tendant vers α sur Γ_α,

$$f'(z) = o\left(\frac{\sqrt{|z-\alpha|}}{1-|z|}\right).$$

p. 962 ligne 9 $\int_{z'_1}^{z_2}$ *lire* $\int_{z'_2}^{z_2}$

p. 963 ligne 8 positis *lire* positifs.